

UNCLASSIFIED

AD 296 210

*Reproduced
by the*

**ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA**



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-2-4

TECHNICAL
REPORT

AS AD 10296210

AS AD 10296210

406

PERTURBATION OF A PLASMA BY A PROBE

JOHN F. WAYMOUTH

TECHNICAL REPORT 406

DECEMBER 26, 1962

296 210

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RESEARCH LABORATORY OF ELECTRONICS
CAMBRIDGE, MASSACHUSETTS

The Research Laboratory of Electronics is an interdepartmental laboratory in which faculty members and graduate students from numerous academic departments conduct research.

The research reported in this document was made possible in part by support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the U. S. Army, the U. S. Navy (Office of Naval Research), and the U. S. Air Force (Office of Scientific Research) under Contract DA36-039-sc-78108, Department of the Army Task 3-99-25-001-08; and in part by Contract DA-SIG-36-039-61-G14.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
RESEARCH LABORATORY OF ELECTRONICS

Technical Report 406

December 26, 1962

PERTURBATION OF A PLASMA BY A PROBE

John F. Waymouth

(Manuscript received June 27, 1962)

Abstract

In the analysis of current-voltage characteristics of a probe in a plasma, as originally developed by Langmuir, it is assumed that the mean-free path of all plasma particles is large in comparison with the probe dimensions, in order that the plasma density, temperature, and potential as derived from the measured characteristics be the same as those of the plasma undisturbed by the probe. When this condition does not hold, the plasma density and potential in the vicinity of the probe are different from those of the undisturbed plasma. The perturbation of the plasma by a probe is analyzed in this report as a problem in ambipolar diffusion, subject to the assumptions that the mean-free path of plasma particles is: (a) comparable to or smaller than probe dimensions, and (b) much greater than the thickness of any sheaths that may be present. Sheaths must therefore be assumed to be thin in comparison with probe dimensions. Analyses are carried through for spherical probes and for plane probes oriented normal to a magnetic field.

The results can be expressed in terms of a parameter Q which is approximately equal at zero sheath potential to the sum of the ratios of probe size to electron mean-free path and probe size to ion mean-free path. Since Q also depends on sheath potential, the current-voltage characteristics are distorted. Methods for determining zero sheath potential, and calculating the properties of the undisturbed plasma from the probe curves are given.

TABLE OF CONTENTS

Glossary	iv
I. Introduction	1
II. Spherical Probe in a Spherical Tube	4
2.1 Perturbation of Electron Density	4
2.2 Perturbation of Plasma Potential	6
III. Disc Probe Normal to a Magnetic Field	9
3.1 Perturbation of Plasma Density by the Probe	9
3.2 Perturbation of Plasma Potential by the Probe	13
IV. Discussion	17
V. Analysis of Previously Published Work	27
Appendix I Mathematical Details Relating to Section II	34
Appendix II Mathematical Details Relating to Section III	37
Appendix III Calculation of Ion Energy at the Sheath Boundary	41
Acknowledgment	42
References	43

GLOSSARY

<u>Symbol</u>	<u>Definition</u>
n	Density of electrons or ions in neutral plasma
n_o	Density of electrons or ions in the unperturbed plasma
n_p	Density of electrons or ions at the sheath boundary (i.e., at the probe)
μ_e, μ_i	Electron, ion mobility
Γ_e, Γ_i	Particle current density of electrons or ions
V_e, V_i	Voltage equivalent of electron, ion temperatures in the plasma $\left(\frac{kT_e}{e}, \frac{kT_i}{e} \right)$
V_{es}, V_{is}	Voltage equivalent of electron, ion temperatures at the sheath boundary
D_a	Ambipolar diffusion coefficient
r_p	Probe radius
V_s	Potential difference across the sheath, positive when probe is positive with respect to plasma outside the sheath
V_o	Potential of the unperturbed plasma
V_p	Potential of the plasma at the sheath boundary (i.e., at the probe)
ΔV	$V_p - V_o$ perturbation of the plasma potential by the probe
γ_e^2	Equals $1 + \mu_e^2 B^2$ (where B is magnetic field)
γ_i^2	Equals $1 + \mu_i^2 B^2$
β^2	Equals $1 + \mu_e \mu_i B^2$
ζ	Equals $z/\beta R$

I. INTRODUCTION

In the analysis of the current-voltage characteristics of a probe in a plasma, as originally developed by Langmuir,¹ it is assumed that the mean-free path of all plasma particles is large in comparison with probe dimensions. When this is the case the electron density and temperature, and plasma potential derived from the analysis are the same as those of the plasma in the absence of the probe.

When this is not the case the plasma density and potential in the vicinity of the probe may be quite different from those of the unperturbed plasma, since ions and electrons must diffuse from the undisturbed regions of the plasma to the vicinity of the probe. If the mean-free path is very short, the density gradients in the plasma may be quite large, and the plasma density in the vicinity of the probe will be substantially different from that of the undisturbed plasma. This situation is aggravated when a magnetic field is present because of the reduction of the diffusion coefficient of plasma particles in the direction transverse to the magnetic field.

The analysis that is given herein was undertaken for the purpose of interpreting the probe characteristics in a discharge tube with an axial magnetic field. Similarly motivated studies have been reported by Bickerton and von Engel² and by Bertotti.³ Neither of these is applicable to the present case. The first study treated plane probes parallel to the magnetic field and was restricted to relatively low magnetic fields, while in the second it was assumed that the longitudinal mean-free path of all particles is long in comparison with any relevant dimension. Neither of these conditions is satisfied in the discharge in which measurements were made.

Because the use of magnetic fields up to 60,000 gauss, or more, was contemplated, the configuration of a plane probe normal to the magnetic field was chosen. We assume the presence of a sheath between probe and plasma. Provided that the sheath is thin in comparison with the mean-free path, the random currents of ions and electrons across the sheath from plasma to probe will be unaffected by the presence of the magnetic field, since they depend only on the component of velocities normal to the probe and parallel to the magnetic field.

Although the probe current for a given plasma density immediately outside the sheath is not affected by the magnetic field, the plasma density and potential themselves most certainly are. The electrons and ions must diffuse to the sheath boundaries, and their diffusion coefficients are strongly affected by the magnetic field.

The problem is one of ambipolar diffusion; we must match the diffusion current from the plasma to the random current across the sheath toward the probe. The axial symmetry of this problem dictated the use of cylindrical coordinates, which has the disadvantage of yielding answers in the form of an infinite series of Bessel functions.

In setting up the problem, we recognized that a similar physical situation exists in the absence of a magnetic field when mean-free paths are small in comparison with probe dimensions. Again, we have to match an ambipolar diffusion current from the plasma

to the random current across the sheath toward the probe. Since this case can be treated in spherical coordinates (as a one-dimensional problem), provides considerable insight into the physical basis of the problem, and yields answers in a particularly simple closed form, it will be treated in some detail first.

The results of this analysis are far from being a trivial warm-up exercise, since many recent probe investigations of normal discharges in the absence of magnetic fields have invaded the domain of probe sizes comparable to or greater than mean-free paths. The results reported herein can be used to estimate the degree of perturbation of the plasma by the probes in these experiments, and permit the correction of the data for the errors introduced thereby.

We note that the procedure of matching diffusion current to random current is similar to the treatment by Allis and Buchsbaum⁴ of the "pre-sheath." This they classify as the region of the plasma (in which $n_i = n_e$) in which the ions collected by the probe are produced, and into which there is a penetration of electric field from the probe to satisfy the condition of continuity of ion current. Our work differs from that of these authors in that they consider ion mean-free paths to be large in comparison with all dimensions, and assume electron density to be everywhere in equilibrium with electrostatic potential. I consider the case in which all mean-free paths are short in comparison with the dimensions of the pre-sheath, so that diffusion equations apply. The pre-sheath is then the region of the plasma perturbed by the probe. Moreover, since results are to apply for positive probe potentials at which electron currents may be large, the assumption that the electron density is in thermal equilibrium with electrostatic potential cannot be made.

A calculation similar to that undertaken here has been reported by Davydov and Zmanovskaja.⁵ These authors consider the case of ambipolar diffusion to a sheath surrounding a spherical or cylindrical probe in the absence of a magnetic field for ion temperatures that are negligible in comparison with electron temperatures. Boyd⁶ has criticized their results on the ground that the boundary conditions at the sheath edge are much more complex than the abrupt discontinuity between neutral plasma and unipolar sheath employed by Davydov and Zmanovskaja. Boyd distinguishes between three regions: (a) a unipolar sheath in which ions are falling freely toward the probe; (b) a quasi-neutral region in which ions are being accelerated toward the probe, making a few collisions with gas atoms, with velocity proportional to the square root of electric field; (c) a diffusion region where ions move by ambipolar diffusion in a neutral plasma.

In the diffusion region the plasma density at the inner boundary is determined in part by the radius of the inner boundary; the results of Davydov and Zmanovskaja agree with those of Boyd when this radius is not sensibly different from that of the probe, that is, when the thickness of sheaths (a) and (b) are small in comparison with probe radius. When this is not true, the problem becomes much more complex because the "effective probe radius" to be used for the ambipolar diffusion part of the problem becomes a function of the potential drop across the sheath, and the more elaborate matching procedure of Boyd must be used.

We note that in high electron density plasmas the thicknesses of the regions (a) and (b) can be small, especially in the range of only a few volts on either side of zero potential drop across them. We shall therefore follow the procedure of Davydov and Zmanovskaja, including the ion temperature explicitly, and show that in the domain of validity of this procedure ($n_e r_p \geq 10^9 \text{ cm}^{-2}$, $r_p/\lambda \geq 0.1$) information about the plasma ion temperature can be deduced from the shape of the probe characteristic near zero sheath potential.

In Section II we shall consider a spherical probe in a spherical discharge tube, following closely the procedure of Davydov and Zmanovskaja. In Section III, a plane probe normal to an axial magnetic field will be taken up. Section IV will be devoted to a discussion of the results, and Section V to recently reported probe studies in the light of these results. Insofar as possible, the stress will be placed on physics, and mathematical details will be given in the appendices.

The following assumptions have been made.

(i) The plasmas are those of diffusion-controlled discharges, in which there is continuous ion production that is proportional to electron density to replace diffusion losses of electrons and ions.

(ii) Electrons have a Maxwellian energy distribution; electron and ion temperatures are assumed to be independent of position, and to be unchanged by the presence of the probe.

(iii) The probe radius is very much smaller than the tube radius (<1 per cent). This insures that losses of electrons and ions to the probe are small in comparison with diffusion losses to the walls; this restriction is necessary if electron temperature is to be unchanged by the presence of the probe.

(iv) Ion mobility in the absence of a magnetic field is negligible in comparison with electron mobility.

(v) Probe radius is much greater than sheath thickness; or alternatively, the outer radius of sheath is essentially the same as that of the probe. Sheath thickness is small in comparison with the mean-free path.

(vi) "Classical" diffusion in a magnetic field is assumed to apply.

II. SPHERICAL PROBE IN A SPHERICAL TUBE

2.1 PERTURBATION OF ELECTRON DENSITY

We must first set up the ambipolar diffusion equation, which we do without making the usual assumption that electron and ion currents are equal. We assume only that electron and ion densities are equal and that $\text{div } \Gamma_e = \text{div } \Gamma_i = nv_i$, the rate of ion production per unit volume. We start with the transport equations

$$\left. \begin{aligned} \Gamma_e &= -n\mu_e E - \mu_e V_e \nabla \cdot n \\ \Gamma_i &= n\mu_i E - \mu_i V_i \nabla \cdot n \end{aligned} \right\} \quad (1)$$

Take the divergence of both sides, set $\nabla \cdot \Gamma_e = \nabla \cdot \Gamma_i = nv_i$, and eliminate E between the two.

$$\left(\frac{1}{\mu_e} + \frac{1}{\mu_i} \right) nv_i = -(V_e + V_i) \nabla^2 n$$

$$\boxed{\nabla^2 n + \frac{nv_i}{D_a} = 0} \quad (2)$$

Here,

$$D_a = \frac{\mu_e \mu_i (V_e + V_i)}{\mu_e + \mu_i} = \mu_i (V_e + V_i),$$

with $\mu_i \ll \mu_e$. This is the same equation that we would have obtained if we had assumed $\Gamma_e = \Gamma_i$ as Schottky⁷ did in the ambipolar diffusion theory, but we derive it under far less restrictive assumptions.

We must solve this equation subject to boundary conditions at the probe of radius r_p centered at $r = 0$, and at the wall $r = R$. At the probe, we have

$$\left. \begin{aligned} \Gamma_e(r_p) &= -n_p \sqrt{\frac{eV_{es}}{2\pi m_e}} \epsilon_e = -n_p \mu_e E - \mu_e V_e \frac{dn}{dr} \Big|_{r_p} \\ \Gamma_i(r_p) &= -n_p \sqrt{\frac{eV_{is}}{2\pi m_i}} \epsilon_i = n_p \mu_i E - \mu_i V_i \frac{dn}{dr} \Big|_{r_p} \end{aligned} \right\} \quad (3)$$

where

$$\begin{aligned} \epsilon_e &= e^{V_s/V_{es}} \quad \text{for } V_s < 0; & \epsilon_i &= 1 & \text{for } V_s < 0 \\ \epsilon_e &= 1 & \text{for } V_s > 0; & \epsilon_i &= e^{-V_s/V_{is}} & \text{for } V_s > 0 \end{aligned}$$

The left-hand side of Eqs. 3 represents the current of electrons and ions across the sheath to the probe, as determined by the potential difference V_s across the sheath and the plasma density at the sheath edge, n_p . The right-hand side represents the transport of electrons and ions out of the plasma to the sheath edge under the combined influence of density gradients and electric fields.

Eliminating E between the two equations, we get

$$\frac{1}{n_p} \frac{dn}{dr} \bigg|_{r_p} = \frac{Q}{r_p}, \quad (4)$$

where

$$Q = Q_e + Q_i$$

$$Q_e = \frac{r_p}{\mu_e(V_e + V_i)} \sqrt{\frac{eV_{es}}{2\pi m_e}} \epsilon_e$$

$$Q_i = \frac{r_p}{\mu_i(V_e + V_i)} \sqrt{\frac{eV_{is}}{2\pi m_i}} \epsilon_i$$

The solution of (2) in spherical coordinates is

$$n = \frac{1}{x} \{A \sin x + B \cos x\}, \quad (5)$$

where $x = \sqrt{\frac{v_i}{D_a}} \cdot r$. When no probe is present, $B = 0$, since the density cannot be infinite at $r = 0$; $A = n_0$ and $\sqrt{\frac{v_i}{D_a}}$ must equal π/R in order that $n = 0$ at the walls.

When a probe is present with radius small enough that $x_p = \pi r_p/R \ll 1$, the value of v_i is not altered by the presence of the probe and x remains equal to $\pi r/R$. We determine the ratio B/A from boundary condition (4) by equating $\frac{1}{n} \frac{dn}{dr} \bigg|_{r_p}$ from (5) to Q/r_p and solving for B/A . It is shown in Appendix II that

$$\frac{B}{A} = -\frac{Qx_p}{1+Q}. \quad (6)$$

If the probe radius is very much smaller than the tube radius, the entire region of the plasma which is perturbed will be that for which n in the absence of the probe is $n = n_0 \frac{\sin x}{x} \approx \frac{n_0 x}{x} = n_0$. In substituting (6) in (5) we can regard $\cos x \approx 1$ and $\frac{\sin x}{x} \approx 1$ and obtain for the density in the perturbed plasma

$$n = n_0 \left\{ 1 - \left(\frac{Q}{1+Q} \right) \frac{x_p}{x} \right\}. \quad (7)$$

The plasma density at the sheath edge, which will be determined from the Langmuir-probe analysis, is

$$n_p = \frac{n_0}{1 + Q}. \quad (8)$$

The plasma density versus distance from the probe given by Eq. 7 is shown in Fig. 1. Note that this is a very slowly varying function. At a distance of 10 probe radii into the plasma the perturbation is still 10 per cent as large as it is at the probe surface. We shall have more to say about this point in Section IV.

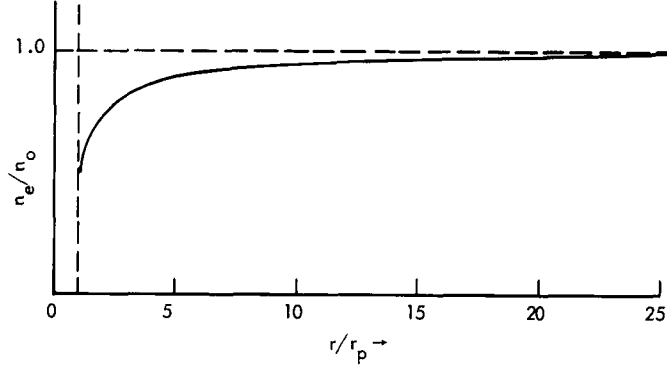


Fig. 1. Electron density in the plasma as a function of distance from a spherical probe for which $Q = 1$.

2.2 PERTURBATION OF PLASMA POTENTIAL

To calculate the electric field, we return to the transport equations, take the divergence of both sides, and solve for the divergence of nE , noting that $\text{div } \Gamma_e = \text{div } \Gamma_i$

$$\nabla \cdot (nE) = - \left(\frac{\mu_e V_e - \mu_i V_i}{\mu_i + \mu_e} \right) \nabla^2 n.$$

Invoking the fact that $\mu_i \ll \mu_e$, and assuming that V_i will be comparable to or less than V_e , we obtain

$$\nabla \cdot (nE) = -V_e \nabla^2 n. \quad (9)$$

In spherical coordinates,

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 n E_r) = -V_e \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial n}{\partial r} \right)$$

$$\int_{r_p}^r \frac{\partial}{\partial r} (r^2 n E_r) dr = -V_e \int_{r_p}^r \frac{\partial}{\partial r} \left(r^2 \frac{\partial n}{\partial r} \right) dr$$

Integrating and solving for the electric field, we obtain

$$E_r = -V_e \frac{1}{n} \frac{dn}{dr} + K \frac{r_p^2}{r^2} \cdot \frac{1}{n} \quad (10)$$

where

$$K = \left[nE + V_e \frac{dn}{dr} \right]_{r_p},$$

and can be evaluated by solving Eq. 3 for E_p , the electric field at the edge of the sheath.

It is shown in Appendix I that $K = \frac{n_o}{r_p} \frac{Q_e(V_e + V_i)}{1 + Q}$.

$$E_r = -V_e \frac{1}{n} \frac{dn}{dr} + \frac{Q_e(V_e + V_i)}{1 + Q} \frac{n_o}{r_p} \frac{1}{nr^2} \quad (11)$$

In the unperturbed plasma, $n \approx$ a constant n_o , and $E \approx 0$. We can regard the potential of the unperturbed plasma as a constant V_o . We shall call the potential of the plasma at the sheath boundary V_p .

$$V_o = V_p - \int_{r_p}^{\infty} E_r dr$$

$$V_p - V_o = \Delta V = \int_{r_p}^{\infty} E_r dr.$$

Here, ΔV is the perturbation of plasma potential at the sheath boundary caused by the presence of the probe.

$$\Delta V = -V_e \int_{r_p}^{\infty} \frac{1}{n} \frac{dn}{dr} \cdot dr + \frac{Q_e(V_e + V_i)}{1 + Q} r_p \int_{r_p}^{\infty} \frac{dr}{r^2 \left(1 - \frac{Qx_p}{x(1+Q)} \right)}$$

The first integral is equal to $-V_e \ln \frac{n_o}{n_p} = -V_e \ln(1+Q)$.

It is shown in Appendix I that the second is equal to $\frac{Q_e(V_e + V_i)}{1 + Q} \ln(1+Q)$. Therefore

$$\Delta V = - \left[V_e - \frac{Q_e(V_e + V_i)}{Q} \right] \ln(1+Q). \quad (12)$$

The first term in (12) is the Boltzmann term, resulting from the fact that the potential and electron density are related in thermal equilibrium according to $n \sim \exp(V/V_e)$.

From this term the plasma potential is shifted negative by the presence of the probe.

The second term is large only when Q_e is a large fraction of Q . Reference to Eq. 4 shows that this can happen only when the sheath potential is slightly negative, zero or positive. Then the electron current to the probe is large and the electric field in the plasma must be reduced from the thermal equilibrium value to permit the increased electron current to flow. Whether or not the electric field actually reverses, and the perturbation of plasma potential becomes positive, depends on the exact value of the ratios of μ_i/μ_e and V_i/V_e .

III. DISC PROBE NORMAL TO A MAGNETIC FIELD

3.1 PERTURBATION OF PLASMA DENSITY BY THE PROBE

The basic procedure is the same as before; set up the ambipolar diffusion equation under the assumption that $\Gamma_i \neq \Gamma_e$, $\text{div } \Gamma_i = \text{div } \Gamma_e = nv_i$. Solve it subject to the boundary condition that diffusion current toward the sheath boundary from the plasma is equal to the random current across the sheath to the probe. This determines electron density as a function of position; then use the transport equations to calculate the electric field that is necessary to bring about the flow of a given probe current.

The derivation of the ambipolar diffusion equation in the presence of a magnetic field is somewhat more complicated. Allis and Buchsbaum⁴ show that when the azimuthal components of current flow are eliminated the transport equations in cylindrical geometry are:

$$\left. \begin{aligned} \Gamma_{ez} &= -n\mu_e E_z - \mu_e V_e \frac{dn}{dz} \\ \Gamma_{er} &= -\frac{n\mu_e E_r}{\gamma_e} - \frac{\mu_e V_e}{\gamma_e} \frac{dn}{dr} \\ \Gamma_{iz} &= n\mu_i E_z - \mu_i V_i \frac{dn}{dz} \\ \Gamma_{ir} &= \frac{n\mu_i E_r}{\gamma_i} - \frac{\mu_i V_i}{\gamma_i} \frac{dn}{dr} \end{aligned} \right\} \quad (13)$$

Here, $\gamma_e^2 = 1 + \mu_e^2 B^2$, $\gamma_i^2 = 1 + \mu_i^2 B^2$. With only the assumption that $\text{div } \Gamma_i = \text{div } \Gamma_e = nv_i$ there are too few equations to eliminate all components of E and Γ . I choose to eliminate, for the present, both components of the electric field. This procedure gives the result

$$\left. \begin{aligned} \frac{1}{\mu_e} \Gamma_{ez} + \frac{1}{\mu_i} \Gamma_{iz} &= -(V_e + V_i) \frac{dn}{dz} \\ \frac{\gamma_e^2}{\mu_e} \Gamma_{er} + \frac{\gamma_i^2}{\mu_i} \Gamma_{ir} &= -(V_e + V_i) \frac{dn}{dr} \end{aligned} \right\} \quad (14)$$

Note that

$$\left. \begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_{er}) &= \text{div } \Gamma_e - \frac{d\Gamma_{ez}}{dz} = nv_i - \frac{d\Gamma_{ez}}{dz} \\ \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_{ir}) &= \text{div } \Gamma_i - \frac{d\Gamma_{iz}}{dz} = nv_i - \frac{d\Gamma_{iz}}{dz} \end{aligned} \right\} \quad (15)$$

By taking the z derivative of the z component of Eq. 14 and operating with the

r component of the divergence on the r equation, with the aid of (15), we can eliminate all but one of the components of the divergence of Γ_i or Γ_e . I choose to retain $d\Gamma_{ez}/dz$. Straightforward differentiation and algebraic manipulation leads to

$$(V_e + V_i) \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + \gamma_i^2 \frac{d^2 n}{dz^2} \right\} + \left(\frac{\gamma_e^2}{\mu_e} + \frac{\gamma_i^2}{\mu_i} \right) n v_i = \frac{\gamma_e^2 - \gamma_i^2}{\mu_e} \frac{d\Gamma_{ez}}{dz}. \quad (16)$$

The left-hand side of (16) is a normal homogeneous diffusion equation with all of the usual solutions obtained by setting it equal to zero. Because of the right-hand side, there is also a particular integral, a function of z and r , which leads to a whole new family of solutions. We restrict ourselves to four limiting cases:

(i) $\frac{d\Gamma_{ez}}{dz} = 0$. The right-hand side of Eq. 16 is zero and it becomes

$$\frac{D_a}{\beta^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + \frac{\gamma_i^2}{\beta^2} D_a \frac{d^2 n}{dz^2} + n v_i = 0, \quad (17)$$

where $D_a = \mu_i(V_e + V_i)$, $\beta^2 = 1 + \mu_e \mu_i B^2$, and we have assumed $\mu_e \gg \mu_i$ throughout. This case would represent conditions near the anode of a discharge tube carrying a current that is independent of z in an axial magnetic field.

(ii) $\frac{d\Gamma_{iz}}{dz} = 0$. From (14) we have $\frac{d\Gamma_{ez}}{dz} = -\frac{\mu_e}{\mu_i} - \mu_e(V_e + V_i) \frac{d^2 n}{dz^2}$ which when substituted in (16) gives

$$\frac{D_a}{\beta^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + D_a \frac{\gamma_e^2}{\beta^2} \frac{d^2 n}{dz^2} + n v_i = 0. \quad (18)$$

This equation was previously derived by Allis⁴ under the assumption that $\Gamma_{iz} = 0$.

(iii) $\frac{d\Gamma_{ez}}{dz} = n v_i$. Since $\text{div } \Gamma_e = \frac{d\Gamma_{ez}}{dz} + \frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_{er}) = n v_i$, this assumption is tantamount to assuming that the radial component of the divergence of the electron current flow is negligible in comparison with both the z component and the production per unit volume.

Under this assumption, Eq. 16 becomes

$$\frac{D_a}{\gamma_i} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + D_a \frac{\partial^2 n}{\partial z^2} + n v_i = 0. \quad (19)$$

Equation 19 was previously derived by Simon.⁷

(iv) $\frac{d\Gamma_{ez}}{dz} = -\frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_{er}) + n v_i$, with both of the terms on the right-hand side being of the same order of magnitude. It is shown in Appendix II that this assumption leads to

$$\frac{D_a}{\beta^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + D_a \frac{\partial^2 n}{\partial z^2} + n v_i = 0. \quad (20)$$

This is the "normal" ambipolar diffusion equation that would be obtained by letting $\Gamma_i = \Gamma_e$.

The question now arises, Which, if any, of these four diffusion equations is applicable to the present case? We can rule out the first two on the ground that neither $\frac{d\Gamma_{ez}}{dz}$ nor $\frac{d\Gamma_{iz}}{dz}$ can be zero for the plane probe normal to the z-direction. We are left with the choice between the Simon and the normal ambipolar equations. Here, the choice rests squarely on the relative magnitudes of $n\nu_i$ and $\frac{1}{r} \frac{\partial}{\partial r} (r\Gamma_{er})$.

For the infinitely long positive column in a magnetic field with diffusion to the walls the only loss mechanism, these two are equal. The Simon diffusion equation was originally derived for a relatively short discharge column in an axial magnetic field with electron loss to conducting end plates being the major loss mechanism. For this case $\frac{1}{r} \frac{\partial}{\partial r} (r\Gamma_{er})$ is negligible in comparison with $\frac{d\Gamma_{ez}}{dz}$ and $n\nu_i$, and it becomes more and more negligible the higher the magnetic field.

We have from the outset restricted ourselves to considering probes that are small in radius in comparison with tube radii. If the probe radius is sufficiently small that electron loss to the probe, even though it may occur from a long column of plasma of radius approximately equal to probe radius, shall be small in comparison with the radial diffusion loss to the tube walls over the "characteristic length" of the positive column, then the electron temperature and ionization frequency in the plasma will be unchanged by the presence of the probe. We therefore employ the normal ambipolar diffusion equation.

It is convenient to express this in dimensionless variables, $\rho = \frac{r}{R}$, $\zeta = \frac{z}{\beta R}$, and let $\nu_i = \left(\frac{2.4}{R}\right)^2 \frac{D_a}{\beta}$. Equation 20 then becomes

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial n}{\partial \rho} \right) + \frac{\partial^2 n}{\partial \zeta^2} + (2.4)^2 n = 0. \quad (21)$$

Equation 21 is to be solved subject to the boundary conditions:

$$\text{for } \zeta = \infty \quad n = n_0 J_0(2.4\rho)$$

At $\zeta = 0$, at the probe, for $r < r_p$,

$$\left. \begin{aligned} -n_p \sqrt{\frac{eV_{es}}{2\pi m_e}} \epsilon_e &= -n_p \mu_e E_z - \mu_e V_e \frac{dn}{dz} \Big|_{z=0} \\ -n_p \sqrt{\frac{eV_{is}}{2\pi m_i}} \epsilon_i &= n_p \mu_i E_z - \mu_i V_i \frac{dn}{dz} \Big|_{z=0} \end{aligned} \right\}. \quad (22)$$

Equations 22 are essentially the same as Eq. 3 and lead to the same boundary condition:

$$\frac{1}{n_p} \frac{dn}{dz} \Big|_{z=0} = \frac{Q}{r_p} \quad \text{for } r < r_p, \quad (23)$$

where Q is as we have defined it in Eq. 4. Since the whole problem is symmetrical about $\zeta = 0$, in the plasma for $r > r_p$ we must have $\frac{dn}{dr} = 0$ at $\zeta = 0$, otherwise the density gradient would have a discontinuity at $\zeta = 0$. Substituting ρ for r in (23), we can summarize the boundary conditions:

$$\left. \begin{aligned} \zeta = 0 \quad \frac{dn}{d\zeta} &= \beta \left(\frac{n_p Q}{\rho_p} \right) & \rho \leq \rho_p \\ \frac{dn}{d\zeta} &= 0 & \rho > \rho_p \\ \zeta = \infty \quad n &= n_o J_o(2.4\rho) \end{aligned} \right\}. \quad (24)$$

It is shown in Appendix II that the solution of (21), subject to conditions (24), is

$$n(\rho, \zeta) = n_o J_o(2.4\rho) - 2\beta Q n_p \sum_{j=2}^{\infty} \frac{\exp(-\sqrt{m_j^2 - (2.4)^2} \zeta) J_1(m_j \rho_p) J_o(m_j \rho)}{m_j \sqrt{m_j^2 - (2.4)^2} J_1^2(m_j)}$$

Here, m_j is the j^{th} zero of J_o , the Bessel function of zero order. To determine the ratio of n_p to n_o , set $\rho = 0$, $\zeta = 0$. The left-hand side then becomes n_p , the plasma density at the sheath edge, by definition. Solve for n_p

$$\frac{n_p}{n_o} = \frac{1}{1 + 2\beta Q \sum_{j=2}^{\infty} \frac{J_1(m_j \rho_p)}{m_j \sqrt{m_j^2 - (2.4)^2} J_1^2(m_j)}} \quad (25)$$

The series is a series of alternating terms of decreasing magnitude, and therefore converges, but the convergence is very slow. It can be greatly improved by calculating $n(\rho=0, \zeta=\epsilon)$, where ϵ is small in comparison with the sheath thickness. This will differ by a negligible amount from the plasma density at the sheath boundary; the retention of the decreasing exponential terms in the series greatly improves the convergence, and in fact, makes it absolute.

Note, however, that in the limit of zero magnetic field, $\beta = 1$, and we are dealing with the same problem that we have treated in Section II, although in a different geometry. Equation 7 shows that the perturbation of plasma density extends many probe radii into the plasma. Figure 2 is intended to suggest that outside of a certain radius, which is somewhat greater than the probe radius, the contours of constant plasma density will be much the same whether the probe is spherical or plane.

Therefore, physical intuition tells us that the perturbation of the plasma by the probe

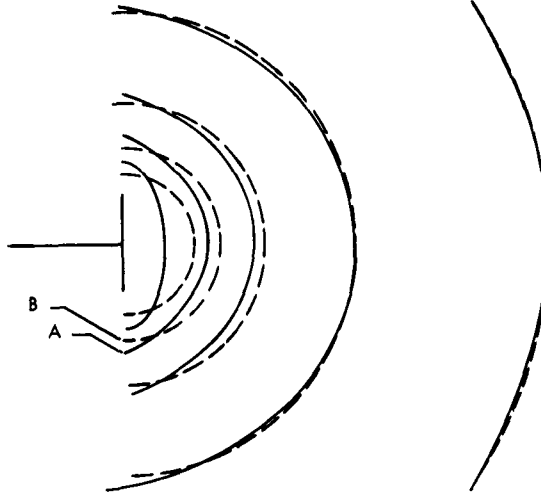


Fig. 2. A: contours of constant electron density in the plasma as a function of distance from a plane probe. B: contours of constant electron density in the plasma as a function of distance from a spherical probe. The difference between A and B is slight except very near the probe.

will be essentially the same whether it is a plane probe or a spherical one of the same radius, within numerical factors of the order of unity. We conclude, then, that the infinite series in Eq. 25, which does not depend on magnetic field, has a sum of approximately $1/2$. For the plane probe in a magnetic field, therefore, Eq. 8 is replaced by

$$n_p = \frac{n_o}{1 + \beta Q} \quad (26)$$

The effect of the magnetic field then is to magnify the Q -value for a particular probe radius - mean-free path combination by the factor $\beta = \sqrt{1 + \mu_e \mu_i \beta^2}$. In order-of-magnitude numbers, this is a factor of 10 at 1 mm pressure and 10,000 gauss.

3.2 PERTURBATION OF PLASMA POTENTIAL BY THE PROBE

The procedure is the same as that in Section II. Calculate the electric field from the transport equations, and integrate from the probe to infinity to find the difference in potential between the plasma at the probe sheath surface and the undisturbed plasma at infinity.

We start with Eqs. 13, and obtain the divergence of Γ_e and Γ_i , setting both equal to $n\nu_i$

$$\text{div } \Gamma_e = n\nu_i = -\mu_e \frac{\partial}{\partial z} (nE_z) - \mu_e V_e \frac{\partial^2 n}{\partial z^2} - \frac{\mu_e}{2} \frac{1}{r} \frac{\partial}{\partial r} (rnE_r) - \frac{\mu_e V_e}{2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right)$$

$$\text{div } \Gamma_i = n\nu_i = \mu_i \frac{\partial}{\partial z} (nE_z) - \mu_i V_i \frac{\partial^2 n}{\partial z^2} + \frac{\mu_i}{2} \frac{1}{r} \frac{\partial}{\partial r} (rnE_r) - \frac{\mu_i V_i}{2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right).$$

Eliminate $\frac{1}{r} \frac{\partial}{\partial r} (rnE_r)$ between the two equations.

$$\begin{aligned} \left(\frac{\gamma_i^2}{\mu_i} + \frac{\gamma_e^2}{\mu_e} \right) nv_i &= \left(\gamma_i^2 - \gamma_e^2 \right) \frac{\partial}{\partial z} (nE_z) - \left(\gamma_e^2 V_e + \gamma_i^2 V_i \right) \frac{\partial^2 n}{\partial z^2} - (V_e + V_i) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) \\ \frac{\gamma_i^2}{\mu_i} + \frac{\gamma_e^2}{\mu_e} &= \frac{\mu_e + \mu_e \mu_i^2 B^2 + \mu_i + \mu_e \mu_i B^2}{\mu_i \mu_e} = \frac{\mu_e (1 + \mu_e \mu_i B^2)}{\mu_e \mu_i} \\ &= \frac{\beta^2}{\mu_i} \\ &= (V_i + V_e) \frac{\beta^2}{D_a}. \end{aligned} \quad (27)$$

Here, we have neglected μ_i in comparison with μ_e . From Eq. 20, we have

$$-(V_e + V_i) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) = \beta^2 (V_e + V_i) \frac{\partial^2 n}{\partial z^2} + \beta^2 \frac{(V_i + V_e)}{D_a} nv_i.$$

Substituting in Eq. 27 and solving for E_z gives

$$\frac{\partial}{\partial z} (nE_z) = - \frac{(\gamma_e^2 V_e + \gamma_i^2 V_i - \beta^2 (V_i + V_e))}{\gamma_e^2 - \gamma_i^2} \frac{\partial^2 n}{\partial z^2}.$$

Straightforward algebraic manipulation shows that this reduces to

$$\frac{\partial}{\partial z} (nE_z) = - \left(\frac{\mu_e V_e - \mu_i V_i}{\mu_e + \mu_i} \right) \frac{\partial^2 n}{\partial z^2}. \quad (28)$$

This is basically the same equation as the one that we obtained for the spherical case, except that here we have z derivatives instead of r derivatives. The fact that the magnetic field drops out as far as the relationship between E_z and dn/dz is concerned is one that we could have deduced by intuition, since plasma particles diffuse freely parallel to B .

Neglecting μ_i in comparison with μ_e and integrating (28) gives

$$E_z = -V_e \frac{1}{n} \frac{\partial n}{\partial z} + \frac{K(r)}{n}. \quad (29)$$

We determine $K(r)$ from the value of E_z , n_p , and dn/dz at $r=0$, $z=0$.

$$K(0) = n_p E_p + V_e \frac{dn}{dz} \Big|_{z=0}$$

Since the boundary conditions for the plane probe at $z=0$ are the same as those for the spherical probe at $r=r_p$, K has the same value as that derived in Appendix I:

$$K(0) = \frac{n_p Q_e (V_e + V_i)}{r_p}.$$

In this case, however, $n_p = \frac{n_o}{1 + \beta Q}$, so that (29) becomes

$$E_z = -V_e \frac{1}{n} \frac{dn}{dz} + \frac{n_o Q_e (V_e + V_i)}{(1 + \beta Q) r_p \cdot n} \quad (30)$$

where this applies only for $r=0$.

Note that the exact integral of this expression from $z=0$ to $z=\infty$ along $r=0$ will yield the perturbation of plasma potential. This will involve integrating over infinite series, which we do not consider worth while now. We shall, therefore, resort to an approximation for the variation of n with z , as shown in Fig. 3.

With this approximation

$$n = \frac{n_o}{1 + \beta Q} \left(1 + \frac{Qz}{r_p} \right) \quad z \leq \beta r_p$$

$$= n_o \quad z \geq \beta r_p$$

and we carry out the integration between the limits of n_p and n_o ($z=0$ to $z=\beta r_p$):

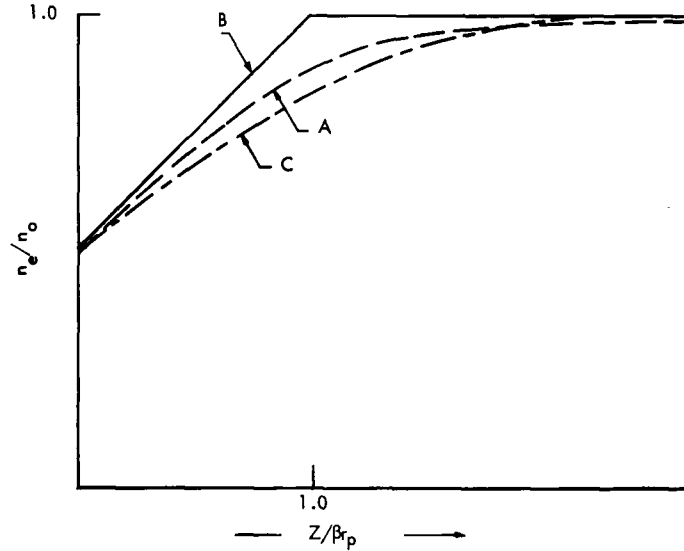


Fig. 3. A: Schematic representation of correct variation of electron density vs distance along the z axis away from a plane probe. B: The function $n_e/n_o = \frac{1}{1 + \beta Q} (1 + Qz/r_p)$ used for the integration of Eq. 30 in Section III. C: The function $n_e/n_o = \left(1 - \frac{\beta Q}{1 + \beta Q} \exp(-z/\beta r_p) \right)$ used for the integration of Eq. 30 in Appendix II.

$$\begin{aligned}
E_z &= -V_e \frac{1}{n} \frac{dn}{dz} + \frac{Q_e(V_e + V_i)}{r_p} \frac{1}{1 + Qz/r_p} \\
\Delta V &= -V_e \ln \left(\frac{n_o}{n_p} \right) + \frac{Q_e(V_e + V_i)}{r_p} \int_0^{\beta r_p} \frac{dz}{1 + Qz/r_p} \\
\Delta V &= -V_e \ln(1 + \beta Q) + \frac{Q_e(V_e + V_i)}{Q} \ln(1 + \beta Q) \\
\Delta V &= - \left\{ V_e - \frac{Q_e(V_e + V_i)}{Q} \right\} \ln(1 + \beta Q) \tag{31}
\end{aligned}$$

In the limit of zero magnetic field this is seen to reduce to the same value as for the spherical case.

It may be objected that the electron density function chosen in Fig. 3 is highly artificial, but it is shown in Appendix III that the function

$$n = n_o \left(1 - \frac{\beta Q}{1 + \beta Q} e^{-z/\beta r_p} \right)$$

(dotted line in Fig. 3) for the second term in the integral yields

$$\frac{\beta Q_e(V_e + V_i)}{(1 + \beta Q)} \ln(1 + \beta Q)$$

which is sensibly different from Eq. 31 only for $Q \ll 1$, for which a diffusion theory will not apply.

IV. DISCUSSION

From a perusal of the equations derived in Sections II and III it is clear that the ion temperature, both in the plasma and at the sheath boundary, affects the results through its appearance in the Q 's. In low-pressure plasmas in which $V_e \gg V_i$, the ion temperature does not affect Q_e . However, the presence of V_{is} (the ion temperature at the sheath boundary) in the exponential of Q_i shows that this quantity will have a major effect upon the exact shape of the probe characteristic for positive sheath voltages.

The question therefore arises: What is the value of ion temperature at the sheath boundary? Since the ions are accelerated toward the sheath by the ambipolar electric fields, they might be expected to arrive at the sheath edge with a higher energy than they possess in the plasma. Tonks,⁸ in fact, derived the relationship, for long mean-free paths and no collisions, that the ions arrive at the sheath boundary with an energy approximately equal to the electron mean kinetic energy. Allis and Buchsbaum⁴ also derive this result in their treatment of the pre-sheath.

These workers have all been interested in the ion saturation current for very negative probe potentials. The only effect of the ion temperature at the sheath boundary is to determine the ion current density. The effect of ion energy on the probe characteristic at positive probe potentials has been ignored. There have been sound reasons for this, of course; after all, the electron current at zero sheath potential is several hundred times the ion current, and the decrease of ion current to zero with positive sheath potentials cannot even be noticed.

In the regime to which the results of this study apply, there is a much greater effect of ion current on the characteristic for positive sheath potentials. As sheath voltage becomes positive and ions are repelled from the probe, $Q_i \rightarrow 0$. Since Q_i is of the same order of magnitude as Q_e , the value of Q decreases substantially as $Q_i \rightarrow 0$. Therefore the plasma density at the sheath boundary, n_p , increases considerably as $Q_i \rightarrow 0$. This is necessarily reflected as an increase in electron current to the probe as the ions are repelled from the probe.

Therefore, although the change in ion current is much too small to measure directly, the change in electron current, because of the retarding of the ions, may be quite large — a factor of 2, or more. Moreover, as will be made evident presently, the shape of the characteristic for positive sheath potentials is considerably affected by the ion temperature at the sheath boundary.

We are unable to use the result of Jonks; the distance over which the ambipolar electric field extends, and over which the ions are accelerated toward the probe, is several probe radii; that is, large in comparison with the ion mean-free path. The ions are therefore making many collisions on their way to the probe and losing energy in large chunks.

In fact, the energy with which the ions arrive at the sheath boundary is just equal to the energy that they had after their last collision plus the energy gained from the

accelerating field in the last free flight, or

$$V_{is} = \left(\frac{V_i - V_g}{2} \right) - E_p \lambda_i,$$

where E_p is the ambipolar electric field at the sheath edge, V_g is the gas temperature, and λ_i is the ion mean-free path. The factor of 2 holds only for equal mass of ion and gas atom. It is shown in Appendix III that this leads to the conclusion that $V_{is} \approx V_i$.

Figures 4, 5, 6, and 7 show calculated probe curves for various gas pressure times probe radius products, calculated for mercury ions in argon, under the assumption that $V_i = 1/30, 1/10, 1/3, 1$ multiplied by V_e , respectively. These curves apply only to probes for which sheath thickness is small in comparison with probe dimension, for which a sharp "break" at plasma potential is expected in the unperturbed case.

Note that all of the curves approach the straight line at negative probe potentials; thus it is indicated that the measurement of electron temperature can still be made. The deviation of the curves from the Maxwellian line as zero sheath potential is approached is the result of the second term in Eq. 12. This is the shift in potential in the positive direction (from the thermal equilibrium potential) required to draw electrons in toward the probe.

The most surprising feature of these curves is the fact that the "knee" occurs not at zero sheath potential but at a positive sheath potential. The reason for this is that electron current continues to increase with positive sheath potentials as ions are repelled, $Q_i \rightarrow 0$ and the electron density at the sheath edge increases from $1/(1+Q_e+Q_i)$ to $1/(1+Q_e)$. Saturation electron current is not achieved essentially until all of the ions are repelled and electron density at the sheath boundary no longer changes with increasing positive sheath potential.

We are led to the conclusion that there is no clear-cut experimental method of identifying zero sheath potential without some additional information. At high ion temperatures, the "knee" disappears completely, while at lower ion temperatures the positive displacement of the "knee" from zero sheath potential depends on the ion temperature. It is necessary to resort to a series of successive approximations and calculated values of the Q 's to be able to identify zero sheath potential from a given set of experimental data.

The Q 's cannot be calculated without knowledge of electron and ion temperatures. The electron temperature can be obtained in the usual way, from the slope of the straight line portion of the logarithmic plot of electron current against probe potential. In order to estimate the ion temperature, the following empirical procedure can be used. Measure the potential difference between the extrapolated electron-current vs probe-potential straight line, and the "knee" of the experimental curve. Since the "knee" may be difficult to identify and the procedure is empirical, it is preferable to measure this potential difference at some lower current than saturation, say 90 per cent of saturation (see Fig. 8). Figure 9 shows this potential difference, ΔV_k vs V_i/V_e for various values of

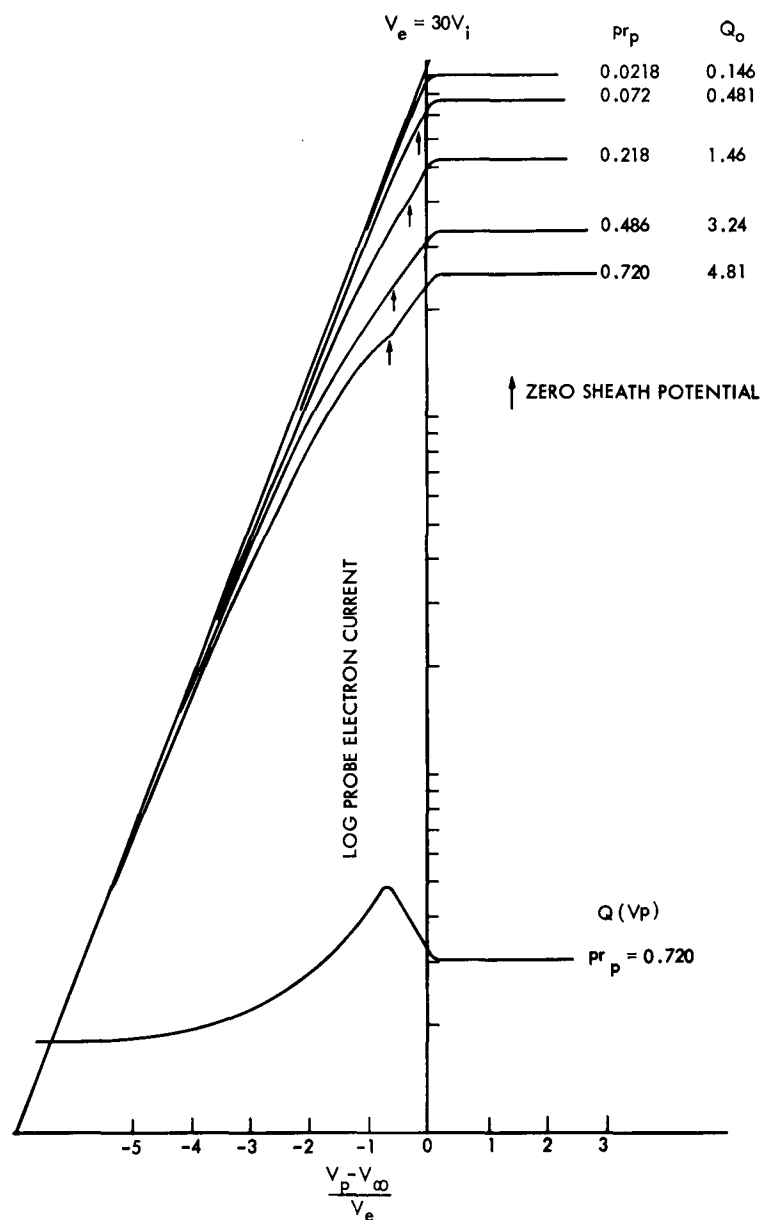


Fig. 4. Probe electron current vs probe-to-plasma potential difference for $V_e = 30 V_i$ and various values of Q ($V_g = 0$). Arrows indicate location of zero sheath potential. Pressure and probe radius products giving those values of Q for mercury ions in argon with 1 ev electron temperature are also given.

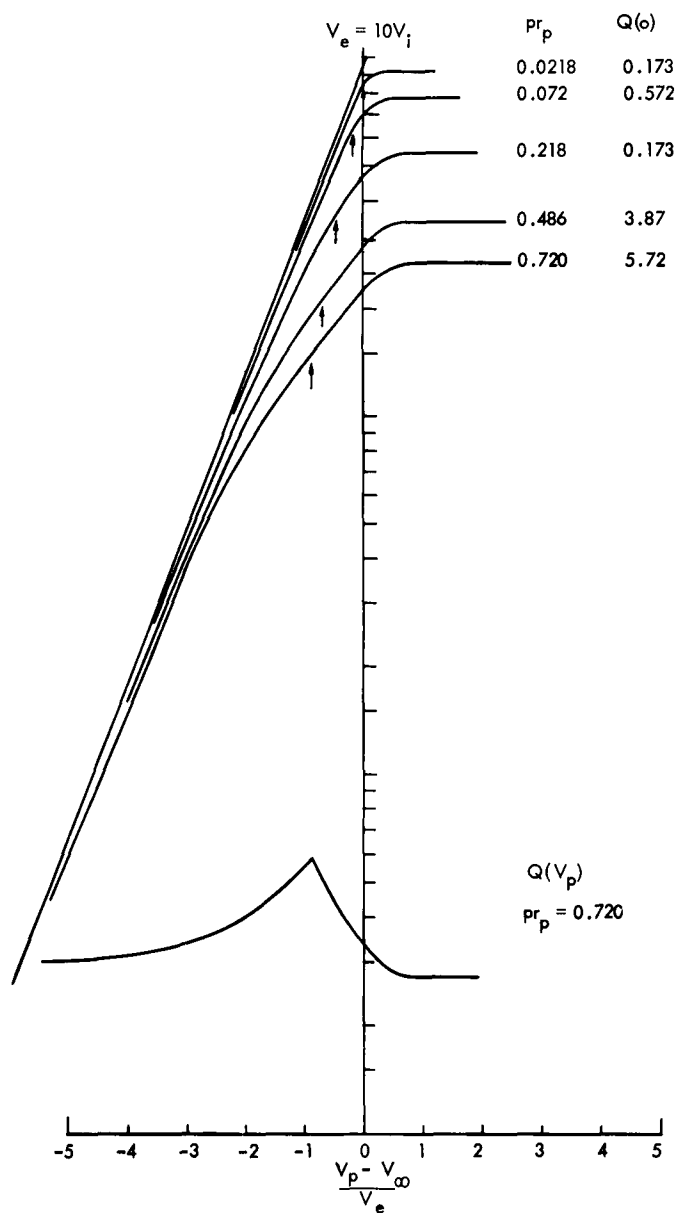


Fig. 5. Probe electron current vs probe-to-plasma potential for $V_e = 10 V_i$, and the same values of pressure times probe radius as in Fig. 4. Arrows indicate location of zero sheath potential.

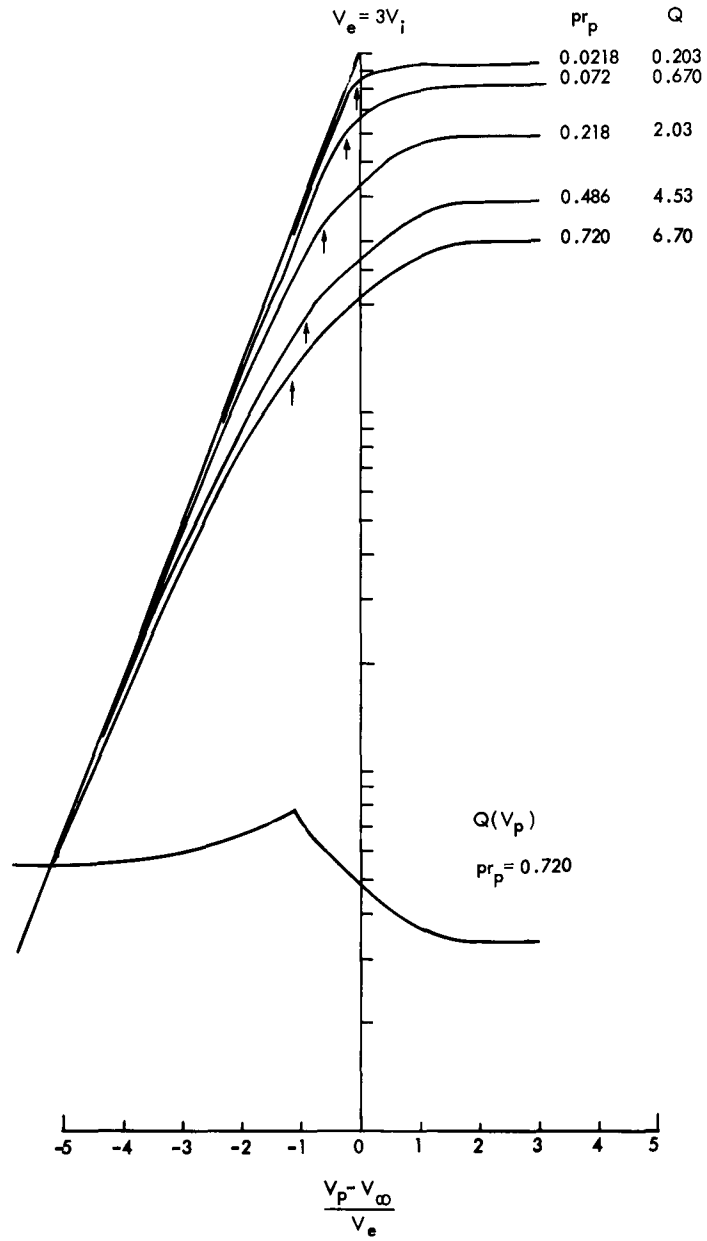


Fig. 6. Probe electron current vs probe-to-plasma potential for $V_e = 3V_i$, and the same values of pressure times probe radius as in Fig. 4. Arrows indicate location of zero sheath potential.

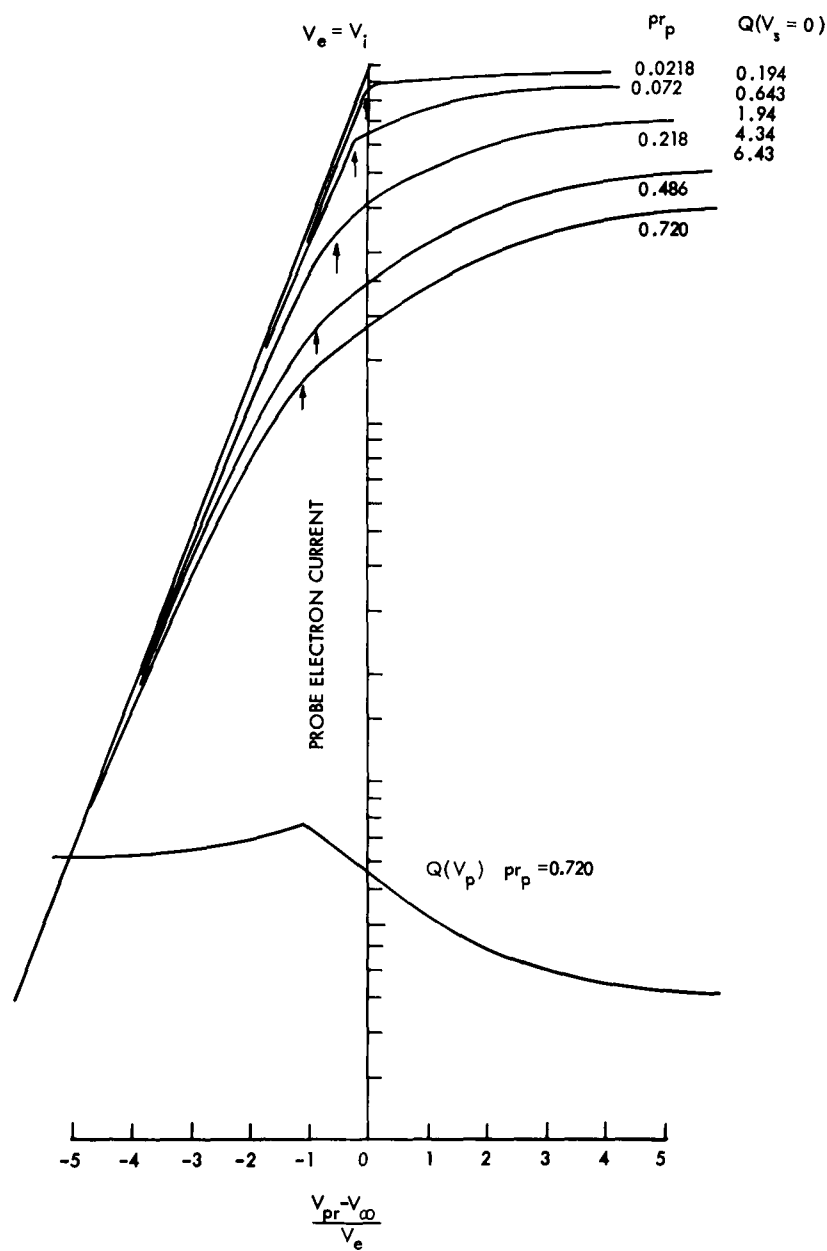


Fig. 7. Probe electron current vs probe-to-plasma potential for $V_e = V_i$, and the same values of pressure times probe radius as in Fig. 4. Arrows indicate location of zero sheath potential.

pressure times probe radius $p \cdot r_p$, under the assumption that there are mercury ions in argon.

From a comparison of an experimental value of ΔV_k with calculated curves of ΔV_k vs V_i/V_e (obtained from calculated curves of electron current vs probe potential) the ratio V_i/V_e can be determined. Hence the value of Q_e and Q_i can be calculated.

Next, using the calculated values of $Q_e (V_s=0)$, $Q_i (V_s=0)$, we determine the electron current for which the potential difference between the extrapolated i_e vs V_p line and the experimental curve is

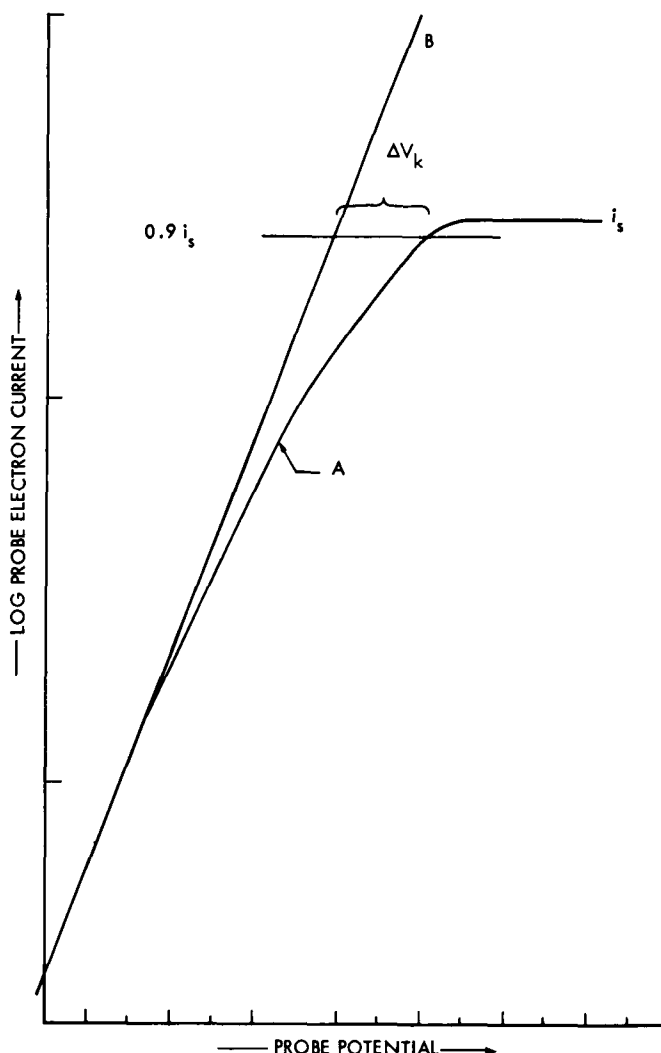


Fig. 8. Construction for obtaining ΔV_k . A, measured electron-current vs probe-voltage characteristic. B, extrapolated linear portion of the characteristic. Draw a line at a current 90 per cent of the saturation current; ΔV_k is the potential difference between the intersections of this line with A and B.

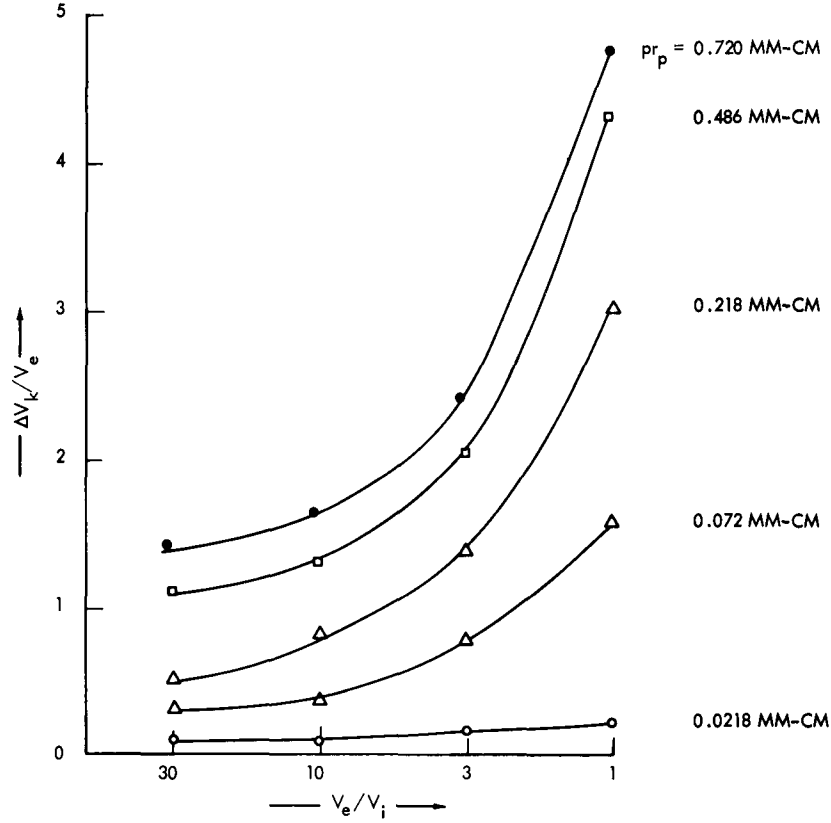


Fig. 9. $\Delta V_k/V_e$ vs V_e/V_i for various values of $\Gamma \cdot r_p$ for mercury ions in argon.

$$\Delta V_1 = \left(\frac{Q_e}{Q_e + Q_i} \right) (V_e + V_i) \ln (1 + Q_e + Q_i) \Big|_{V_s = 0}$$

(See Fig. 10.) This is the electron current for $V_s = 0$. Then, extrapolate the i_e vs V_p line for a potential difference, ΔV_2 , beyond this current.

$$\Delta V_2 = V_e \ln (1 + Q_e + Q_i) \Big|_{V_s = 0}$$

The current at this potential is the random current density in the undisturbed plasma. The probe potential at this point is the potential of the undisturbed plasma, and the potential difference between this point and the point of zero sheath potential is given by Eq. 12:

$$\Delta V = - \left(V_e - \frac{Q_e}{Q_e + Q_i} (V_e + V_i) \right) \ln (1 + Q_e + Q_i) \Big|_{V_s = 0}$$

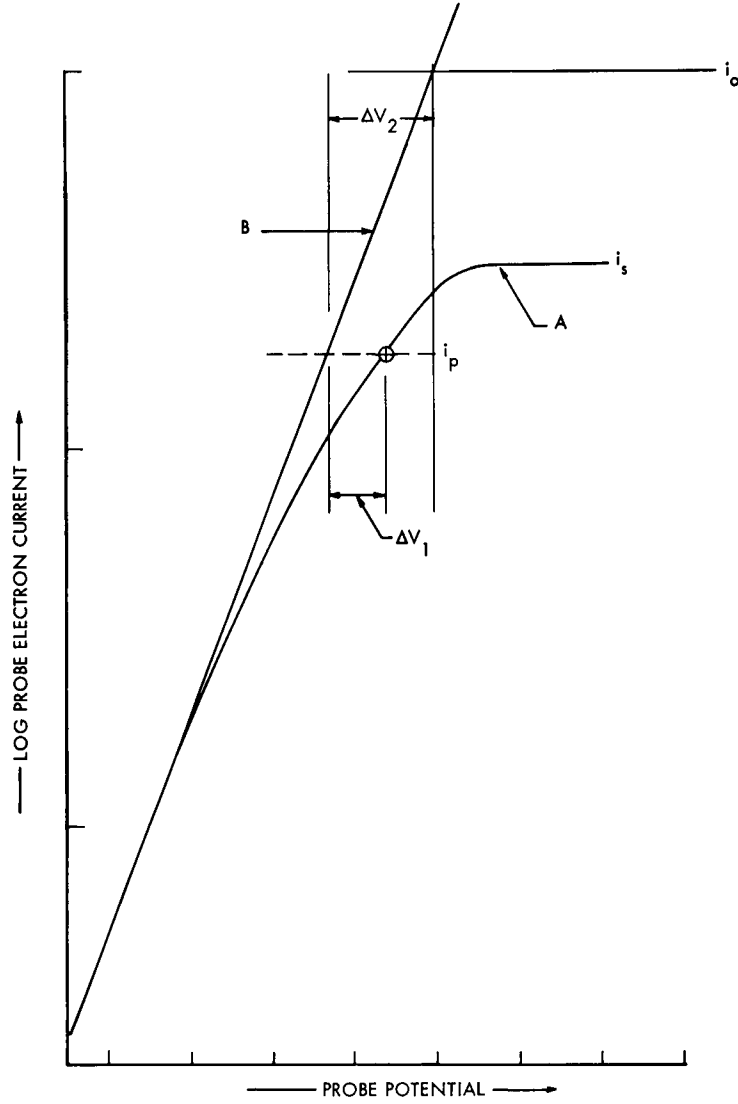


Fig. 10. Construction for determining zero sheath potential. A and B as in Fig. 8.

Find the current for which $\Delta V_1 = \frac{Q_e(V_e + V_i)}{Q_e + Q_i} \ln(1 + Q_e + Q_i) \Big|_{V_s=0}$. Intersection

of this current with A is at zero sheath potential \oplus . Unperturbed plasma potential is ΔV_2 volts positive from the intersection of this current with B, where $\Delta V_2 = V_e \ln(1 + Q_e + Q_i) \Big|_{V_s=0}$. Current at which B intersects unperturbed plasma potential is the random current to a probe of the same area in an unperturbed plasma. As a check on consistency, the currents i_o , i_s , i_p should be in the ratio $i_o : i_s : i_p = 1 : \frac{1}{1 + Q_e} : \frac{1}{1 + Q_e + Q_i} \Big|_{V_s=0}$.

As a check on the correctness of the results, the three currents i_o , i_s , i_p ought to be in the ratio

$$i_o : i_s : i_p = 1 : \frac{1}{1 + Q_e} : \frac{1}{1 + Q_e + Q_i} \bigg|_{V_s = 0} .$$

If they are not, the most likely source of error is in the determination of i_s , and therefore of $0.9i_s$, and thus of V_i . Especially at high ion temperatures, the curve will not reach saturation until relatively high positive sheath potentials are reached, and ionization by electrons in the sheath will cause the measured current to increase with increasing probe potential, and the experimental curve never flattens off. Therefore, make an appropriate adjustment in the assumed value of i_s , redetermine the ion temperature, recalculate the Q 's, relocate zero sheath potential, and repeat the entire process until it is self-consistent.

V. ANALYSIS OF PREVIOUSLY PUBLISHED WORK

The purpose of this section is to examine a limited number of recently published high-caliber probe studies in the light of the equations derived in this report. No attempt is made to make a complete survey, nor is there any intention to deprecate fine pieces of work.

The works considered are those of Anderson,⁹ Bills, Holt, and McClure,¹⁰ Medicus,¹¹ Verweij,¹² and Waymouth.¹³ Of these, Medicus employed a spherical probe and Waymouth a plane probe. The others used wire probes. Since wire probes have not been analyzed in the present study, we shall first consider the definition of an "effective radius" for a wire probe. This may be defined as the radius of the spherical probe that causes the same perturbation in plasma density.

It seems reasonable, in the light of the physical mechanism that has been considered, that two probes that draw the same current from a given plasma will cause the same degree of perturbation. Since probes draw current in approximate proportion to their areas, we can say that the cylindrical probe has an effective radius equal to the radius of the sphere with the same surface area:

$$r_{\text{eff}} = \sqrt{\frac{r_p \cdot \ell}{2}},$$

where r_p is the radius, and ℓ is the length of the cylindrical wire probe. Undoubtedly, this is only approximately true, but to improve on it would require a good deal more calculation than seems to be warranted now.

A second point to consider in discussing the reported probe studies is the fact that probe lead-in wires, or supports, are always shielded to prevent collection of electrons or ions. Since shields are frequently of substantially larger radius than the probe itself, especially for wire probes, diffusion to the shield may reduce the plasma density substantially in the vicinity of the shield. This is particularly a problem with wire probes, for which a substantial fraction of the probe area may lie within the perturbed region.

Customarily, the probe shield operates at floating potential; the value of Q for it is therefore equal to Q_i . By calculating Q_i for the end of the shield and regarding it as a spherical probe, we can use Eq. 7 to find the variation of plasma density as a function of distance from the shield (see Fig. 11). We can therefore integrate Eq. 7 to find the average electron density seen by the probe, assuming that the Q for the probe itself is small. For a wire probe, this is given by

$$\frac{n_p}{n_o} = 1 - \frac{Q_{\text{ish}}}{(\ell/r_{\text{sh}} - 1)(1 + Q_{\text{ish}})} \ln(\ell/r_{\text{sh}})$$

where ℓ is the probe length, r_{sh} the shield radius, and Q_{ish} the Q_i of the shield. It

Table I. Perturbations caused by probes used by several authors.

Type of probe	Anderson	Indicus	Verweij	Bills, Holt and McClure	Waymouth
Probe radius	wire 1.25×10^{-3} cm	spherical 2.5×10^{-2} cm	wire 1×10^{-3} cm	wire 4×10^{-2} cm	plane 2.54×10^{-2}
Probe length	0.4 cm	-	0.25 cm	0.8 cm	-
Effective probe radius	1.6×10^{-2} cm	2.5×10^{-2}	1.12×10^{-2}	1.24×10^{-2}	2.54×10^{-2}
Shield radius	0.5 cm	*	1.3×10^{-2} cm	*	7.5×10^{-2}
Gas pressure	3-12 mm Hg	1 mm Ne	0.1-20 mm A	$\sim 2 \times 10^{-3}$ mm Hg	1 mm Ne
Electron temperature	$\sim 500^\circ\text{K}$	$\sim 11,600^\circ\text{K}$	$\sim 11,600^\circ\text{K}$	$\sim 11,600^\circ\text{K}$	$\sim 11,600^\circ\text{K}$
Ion temperature	$\sim 300^\circ\text{K}$	$\sim 300^\circ\text{K}$	$\sim 300^\circ\text{K}$	$\sim 300^\circ\text{K}$	$\sim 300^\circ\text{K}$
$Q_e (V_s = 0)$	0.4-1.63	0.105	$4.7 \times 10^{-3} - 0.94$	0.001	0.106
$Q_i (V_s = 0)$	1.43-5.71	0.076	$2.4 \times 10^{-3} - 0.49$	0.00075	0.056
Q_i (shield)	9-36	-	$2.9 \times 10^{-3} - 0.59$	-	0.167
n_p/n_o at probe because of shield	0.58-0.55	-	1.0-0.94	-	0.86

* Not given by author.

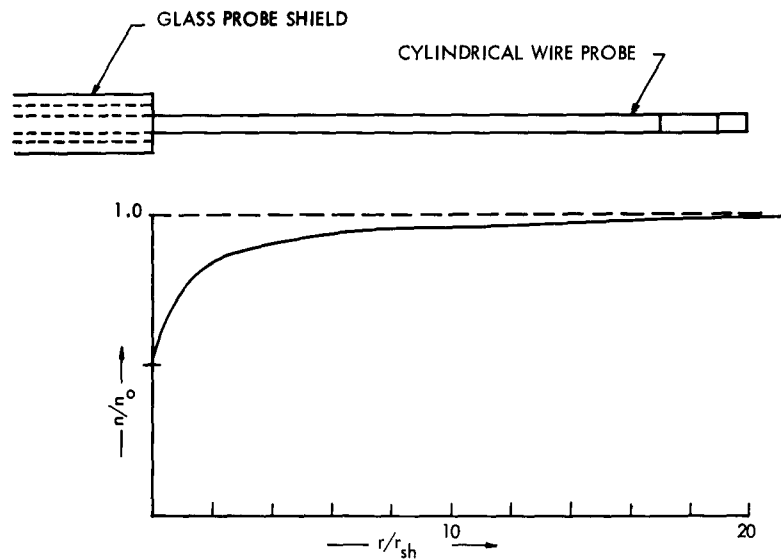


Fig. 11. Electron density in the plasma vs distance from the probe shield for a cylindrical wire probe. Q for the probe itself is assumed negligible, and Q_{ish} is unity.

is plain that for probes that are long in comparison with shield radius, the value of n_p approaches n_0 .

When the Q 's of probe and shield are comparable, and the probe is not long in comparison with the shield radius, the whole diffusion problem becomes much more complicated and the calculation of the corrections to be used with wire probes is dubious. Finally, for the majority of these cases, assumption 5 (Section I) is not really satisfied, and the corrections calculated here can be regarded as valid to order of magnitude only.

Table I shows various pertinent data from the authors cited, together with the calculated values of the Q 's.

It is plain that of the five references cited, only the work of Medicus and of Bills, Holt, and McClure appears to avoid density and potential perturbations. It must be pointed out that a diffusion theory is not expected to apply when either of the Q 's is less than 0.1. Since Q_e , Q_i are approximately equal to the ratio of electron or ion mean-free path to probe radius, and the perturbation in density extends approximately 10 probe radii into the plasma, for Q_e , $Q_i < 0.1$, the entire perturbation would take place in a distance of one mean-free path or less.

Varweij has employed probe measurements to measure electron density, axial electric field, and electron temperature in a discharge in mercury vapor and argon. Figure 12 shows his published data for electron density as a function of argon pressure, together with a corrected value that I have estimated. Since Varweij's measurements apply to the case for which $V_i \ll V_e$, a well-defined saturation current would be expected, and the saturation current would yield $n_p = n_0/(1+Q_e)$. It is also possible to estimate too high

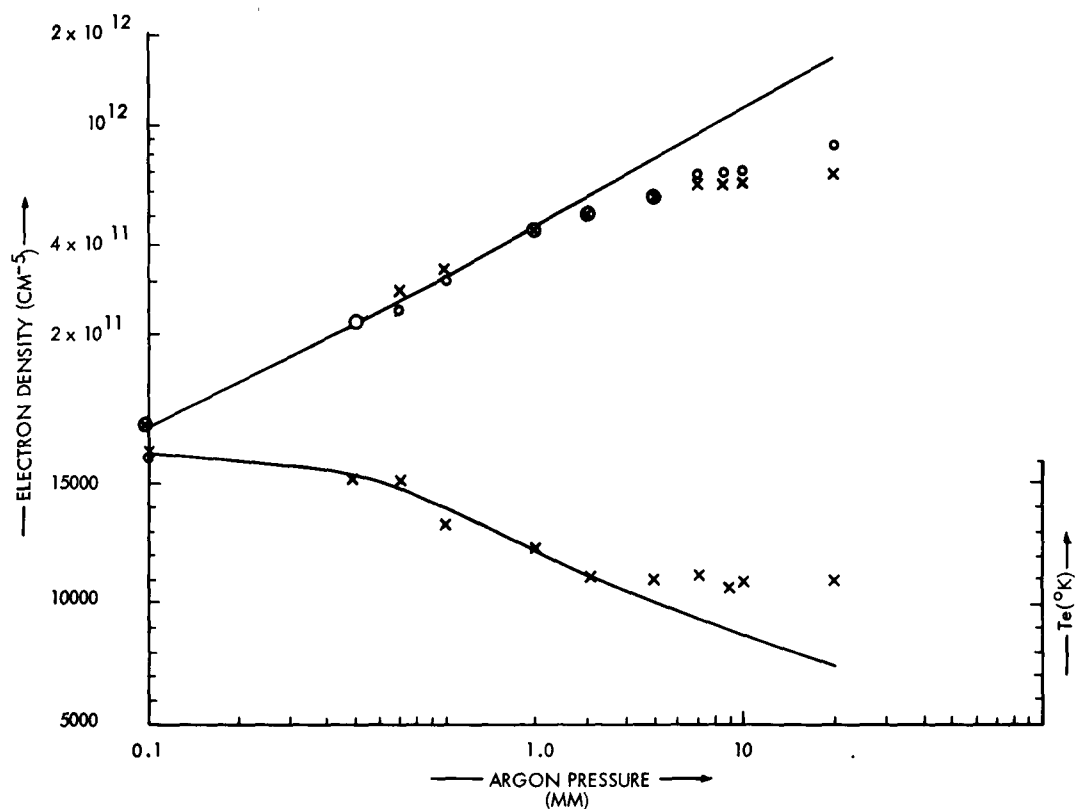


Fig. 12. Data from Verweij¹¹ on electron density and temperature in a discharge at 400 ma in mercury vapor at 7 microns plus argon vs argon pressure (experimental points), together with estimated corrections for plasma perturbation (solid lines).

an electron temperature, if the "best possible straight line" is drawn through the experimental data right up to the apparent zero of sheath potential, instead of relying only on the very negative sheath voltage range. Figure 12 also includes Verweij's data on electron temperature as a function of argon pressure, together with an estimated correction.

The work of Anderson is worthy of special note because Anderson compared probe measurements with microwave electron density measurements and found wide discrepancies between them. Anderson recognized the existence of serious perturbation effects in his work and corrected the probe data according to the method of Davydov and Zmanovskaja. When account was taken of the influence of Coulomb collisions on the diffusion coefficients the agreement between corrected probe values and microwave values for electron density was satisfactory at high electron densities, as shown in Fig. 13. Also shown are the ratios n_p/n_o calculated from the Q 's for various gas pressures with Coulomb scattering neglected. It must be pointed out that, strictly speaking, neither the theory of Davydov and Zmanovskaja nor the present theory ought to apply to the case studied by Anderson, since both assume a production of ionization proportional to the electron density. Anderson's discharge was a glow discharge in which the majority

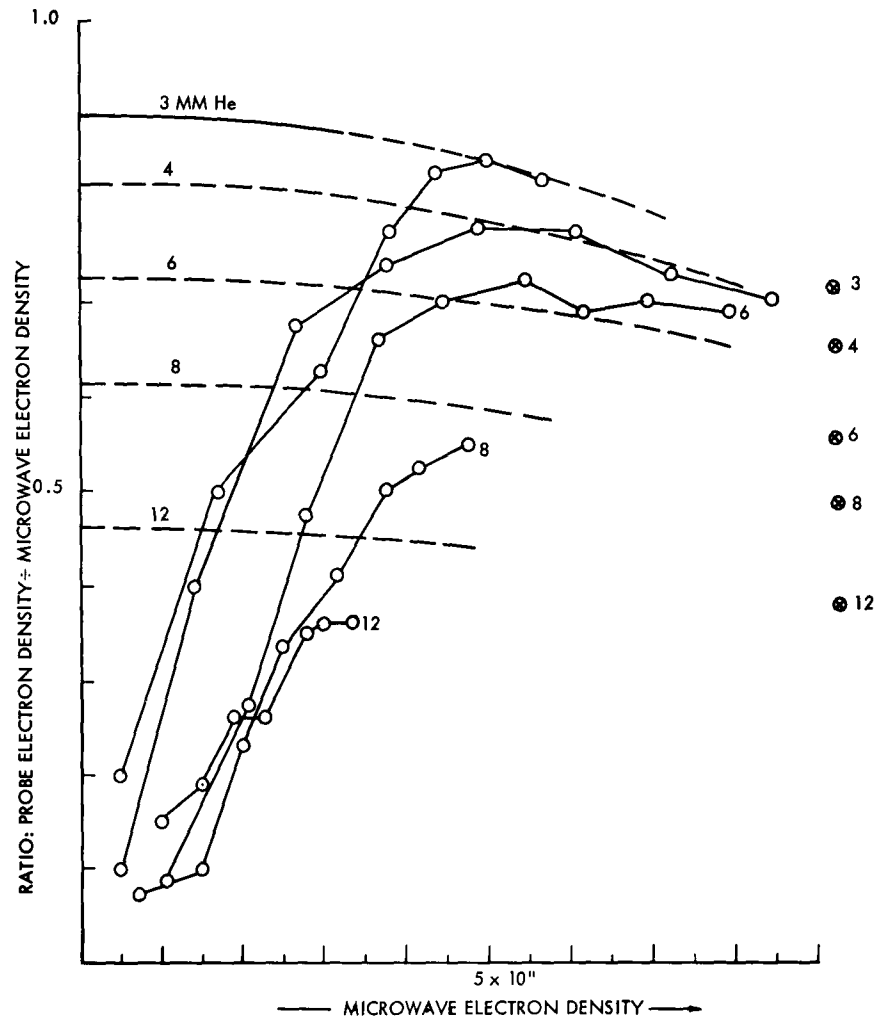


Fig. 13. Data from Anderson⁸ on electron density as measured by probes vs electron density as measured by microwaves for a cold-cathode discharge in helium at various pressures. Also shown are Anderson's calculations on the expected perturbation according to the theory of Davydov and Zmanovskaja, including the effects of Coulomb collisions (---) and the expected perturbation according to these results, neglecting Coulomb collisions but including the fact that $V_e \approx V_i$ (\otimes). For the interpretation of the results at low electron densities refer to Fig. 14.

of the electrons were of very low energy ($T_e \sim 500^\circ\text{K}$) and the ionization was produced mainly by a very few primary electrons of several hundred volts energy and by a somewhat larger number of secondary electrons with $T_e \sim 20,000^\circ\text{K}$. A somewhat different theory in which production is assumed to be independent of position would be more applicable to Anderson's case.

At low electron densities, Anderson regarded his measured electron temperatures as being too high, which fact he ascribed to electric fields from the probe accelerating

electrons in toward the probe and increasing their energy. Too high an electron temperature would lead to too low a calculated electron density to give the measured saturation current density.

Equation 11 indicates, however, that for negative sheath potentials the electric field in the plasma is still retarding for electrons up to zero sheath potential. A more likely explanation for too high an electron temperature is suggested by the type of probe characteristic observed by Anderson,¹⁴ shown in Fig. 14. The presence of a large electron-current tail, which is due to the presence of the relatively high-energy secondary electrons, restricts the range of currents over which the temperature of the ultimate electrons can be measured to the top order of magnitude or so. Figures 6 and 7 show that the estimated electron temperature can easily be too high by a factor of from 3 to 10.

Another point to be noted is that in the internal report¹⁴ from which his journal article was taken, Anderson noted that the electron density as measured from the

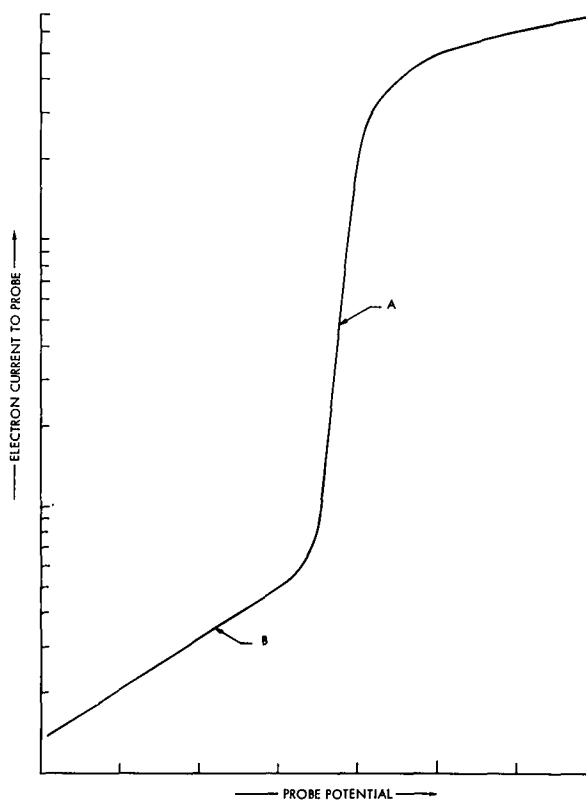


Fig. 14. Schematic of probe electron-current vs probe-potential curves observed by Anderson.¹³ Two-component electron energy distributions are observed: A, "ultimate" electrons of very low temperature; B, secondary electrons ($\sim 50,000^\circ\text{K}$). Comparison with Fig. 4 shows that the electron temperature of the ultimate electrons will be measured from the range of probe potentials near zero sheath potentials, for which the apparent electron temperature will be too high.

electron current was approximately 11 times as great as the ion density measured from the ion current. Since in the plasma $n_i = n_e$, the apparent difference must lie in differences in the degree of perturbation. From the differences between Q_e and Q_i given in Table I, we would expect an apparent electron density approximately 4 times as great as the apparent ion density. It is possible that the effect of the shield could influence the results in the ion-collecting range but have smaller influence in the electron-collecting range, and thus further accentuate the difference.

The work of Bills, Holt, and McClure, and that of the author have in common the use of a pulse technique for making probe measurements. Perturbation effects will be slightly different under these conditions, since the pulses are too fast for the density distribution in the perturbed plasma to change during the pulse. The density distribution in the plasma is fixed at that determined by whatever bias is placed on the probe between pulses. All that changes during the pulse is the electric field in the plasma. It is easy to show that under these conditions, and under the assumption that the probe is biased strongly negative between pulses, that

$$\frac{n_p}{n_o} = \frac{1}{1 + Q_i}$$

$$\Delta V = - \left[V_e - \frac{Q_e (V_e + V_i)}{2(1 + Q_i)} \right] \ln (1 + Q_i).$$

Bills, Holt, and McClure comment that their electron density as measured by pulse methods is 16 per cent, and their electron temperature is 10 per cent greater than those measured by dc methods. While it is true that the values of Q calculated for their discharge are negligibly small, their work suffers from depletion effects of a different sort, since the effective probe radius was 5 per cent of the tube radius, and the saturation electron currents were as great as 20 per cent of the tube current. It seems reasonable to suppose that a different sort of depletion theory, derived for this particular case, would show similar differences between perturbations observed with pulse and dc methods of measurement.

APPENDIX I

MATHEMATICAL DETAILS RELATING TO SECTION II

1. Boundary Condition on Electron Density at the Probe

From Eq. 3,

$$n = \frac{1}{x} \{A \sin x + B \cos x\} \quad x = \frac{\pi r}{R},$$

and from Eq. 4,

$$\left. \frac{1}{n} \frac{dn}{dr} \right|_{r_p} = \frac{Q}{r_p},$$

determine the ratio B/A.

From Eq. 5,

$$\frac{1}{n} \frac{dn}{dr} = \frac{\pi}{R} \frac{1}{n} \frac{dn}{dx} = \frac{\pi}{R} \left\{ \frac{A \cos x - B \sin x}{A \sin x + B \cos x} \right\} - \frac{\pi}{R} \frac{1}{x}.$$

Equating this to (4), noting that $x_p \ll 1$, $\cos x_p = 1$, $\sin x_p = x_p$, we obtain

$$\frac{\pi}{R} \left\{ \frac{A - Bx_p}{Ax_p + B} \right\} - \frac{\pi}{Rx_p} = \frac{Q}{r_p} = \frac{\pi}{R} \frac{Q}{x_p}$$

$$\frac{A - Bx_p}{Ax_p + B} = \frac{1 + Q}{x_p}$$

$$Ax_p - Bx_p^2 = Ax_p(1+Q) + B(1+Q).$$

Since $x_p^2 \ll 1$,

$$\frac{B}{A} = -\frac{Q}{1+Q} x_p.$$

2. Determination of Constant K in Eq. 10

$$K = \left[nE + V_e \frac{dn}{dr} \right]_{r_p} = n_p E_p + V_e \left. \frac{dn}{dr} \right|_{r_p}$$

This we evaluate from the probe boundary condition (Eq. 3) by solving for E_p , the electric field in the plasma at the sheath boundary. Subtracting the two equations and substituting from Eq. 4 the values Q_e and Q_i yields

$$-\frac{n_p Q_e (V_e + V_i)}{r_p} + \frac{n_p Q_i (V_e + V_i)}{r_p} = -2n_p E_p - (V_e - V_i) \left. \frac{dn}{dr} \right|_{r_p}.$$

Subtract $(V_e + V_i) \frac{dn}{dr_p}$ from both sides:

$$-\frac{n_p Q_e (V_e + V_i)}{r_p} + \frac{n_p Q_i (V_e + V_i)}{r_p} - (V_e + V_i) \frac{dn}{dr} \Big|_{r_p} = -2 \left(n_p E_p + V_e \frac{dn}{dr} \Big|_{r_p} \right).$$

For $(V_e + V_i) \frac{dn}{dr} \Big|_{r_p}$ on the left-hand side, substitute from Eq. 4 its value $\frac{n_p Q_e}{r_p} (V_e + V_i) + \frac{n_p Q_i}{r_p} (V_e + V_i)$.

$$K = \left(n_p E_p + V_e \frac{dn}{dr} \Big|_{r_p} \right) = \frac{n_p Q_e (V_e + V_i)}{r_p}.$$

From Eq. 8 substitute the value of n_p . Then

$$K = \frac{n_o Q_e (V_e + V_i)}{(1+Q)r_p}.$$

3. Evaluation of the Integral

$$I = \frac{Q_e (V_e + V_i)}{1+Q} r_p \int_{r_p}^{\infty} \frac{dr}{r^2 \left(1 - \frac{Q x_p}{(1+Q)x} \right)}.$$

Changing variables from r to $x = \frac{\pi r}{R}$ gives

$$I = \frac{Q_e (V_e + V_i)}{1+Q} x_p \int_{x_p}^{\infty} \frac{dx}{x^2 \left(1 - \frac{Q x_p}{(1+Q)x} \right)}.$$

Now change variables again to $u = \frac{x}{x_p}$. Then

$$I = \frac{Q_e (V_e + V_i)}{1+Q} \int_1^{\infty} \frac{du}{u^2 - \frac{Q}{1+Q} u}.$$

The integral can be broken up by partial fractions:

$$I = \frac{Q_e(V_e + V_i)}{1 + Q} \left\{ \frac{1 + Q}{Q} \int_1^\infty \frac{du}{u - \frac{Q}{1 + Q}} - \frac{1 + Q}{Q} \int_1^\infty \frac{du}{u} \right\}$$

$$= \frac{Q_e(V_e + V_i)}{Q} \left\{ \ln \left(\frac{u - Q/(1 + Q)}{u} \right) \right|_1^\infty \right\}$$

$$= \frac{Q_e(V_e + V_i)}{Q} \left\{ -\ln \left(1 - \frac{Q}{1 + Q} \right) \right\}$$

$$= \frac{Q_e(V_e + V_i)}{Q} \ln(1 + Q).$$

APPENDIX II

MATHEMATICAL DETAILS RELATING TO SECTION III

1. Derivation of the Ambipolar Diffusion Equation when $\frac{1}{r} \frac{\partial}{\partial r} (r \Gamma_{er}) \sim n \nu_i$

We assume that $\Gamma_{er} = -D_{a\perp} \frac{\partial n}{\partial r}$, but, for the moment, do not specify what $D_{a\perp}$ is:

$$\frac{\partial \Gamma_{ez}}{\partial z} = n \nu_i + D_{a\perp} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right),$$

and substitute it in Eq. 16.

$$(V_e + V_i) \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + \gamma_i^2 \frac{\partial^2 n}{\partial z^2} \right\} + \left(\frac{\gamma_e^2}{\mu_e} + \frac{\gamma_i^2}{\mu_i} \right) n \nu_i = \left(\frac{\gamma_e^2 - \gamma_i^2}{\mu_e} \right) n \nu_i + \left(\frac{\gamma_e^2 - \gamma_i^2}{\mu_e} \right) D_{a\perp} r \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right)$$

$$(V_e + V_i) \left\{ \left(1 - \frac{\gamma_e^2 - \gamma_i^2}{\mu_e} \frac{D_{a\perp}}{(V_e + V_i)} \right) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + \gamma_i^2 \frac{\partial^2 n}{\partial z^2} \right\} + \frac{\gamma_i^2}{\mu_i} n \nu_i = 0$$

$$\left\{ \frac{\mu_i (V_e + V_i)}{\gamma_i^2} - \frac{\gamma_e^2 - \gamma_i^2}{\gamma_i^2} \frac{\mu_i}{\mu_e} D_{a\perp} \right\} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + \mu_i (V_e + V_i) \frac{\partial^2 n}{\partial z^2} + n \nu_i = 0$$

The term in front of $\frac{\partial^2 n}{\partial z^2}$ is, of course, $D_{a\parallel}$, while the term multiplying $\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right)$ is $D_{a\perp}$.

$$D_{a\perp} = \frac{\mu_i (V_e + V_i)}{\gamma_i^2} - \frac{\gamma_e^2 - \gamma_i^2}{\gamma_i^2} \frac{\mu_i}{\mu_e} D_{a\perp}.$$

Solving for $D_{a\perp}$, and noting that $\mu_i (V_e + V_i)$ is equal to D_a , we have

$$D_{a\perp} = \frac{\mu_e D_a}{\gamma_i^2 \mu_e + \gamma_e^2 \mu_i - \gamma_i^2 \mu_i}.$$

Multiplying out $\gamma_i^2 = 1 + \mu_i^2 B^2$, $\gamma_e^2 = 1 + \mu_e^2 B^2$, and noting that $\mu_i \ll \mu_e$ gives

$$D_{a\perp} = \frac{D_a}{1 + \mu_e \mu_i B^2} = \frac{D_a}{\beta^2}.$$

Therefore

$$\frac{D_a}{\beta^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial n}{\partial r} \right) + D_a \frac{\partial^2 n}{\partial z^2} + n \nu_i = 0$$

which is Eq. 20.

2. Solution of Eq. 21 Subject to Boundary Conditions (24)

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial n}{\partial \rho} \right) + \frac{\partial^2 n}{\partial \zeta^2} + (2.4)^2 n = 0.$$

Assume that n is a separable function of position

$$n = g(\rho) f(\zeta)$$

$$\frac{1}{\rho g} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial g}{\partial \rho} \right) + \frac{1}{f} \frac{\partial^2 f}{\partial \zeta^2} + (2.4)^2 = 0.$$

Let $\frac{1}{f} \left(\frac{\partial^2 f}{\partial \zeta^2} \right) = m_j^2 - (2.4)^2$, for which we shall presently place a restriction on the m_j .

$$f(\zeta) = A_j \exp \left(\sqrt{m_j^2 - (2.4)^2} \zeta \right) + B_j \exp \left(- \sqrt{m_j^2 - (2.4)^2} \zeta \right)$$

In order for n to be finite for $\zeta = \infty$, all A_j 's must be zero. Since $g(\rho) = J_0(m_j \rho)$,

$$n(\rho, \zeta) = \sum_{j=1}^{\infty} B_j \exp \left(- \sqrt{m_j^2 - (2.4)^2} \zeta \right) J_0(m_j \rho).$$

This must be used as a Fourier type of series to match boundary conditions at the probe. Note that

$$\begin{aligned} \int_0^1 \rho \{ J_0(m_j \rho) J_0(m_k \rho) \} \partial \rho &= 0 & \text{if } j \neq k \\ &= \frac{1}{2} J_1^2(m_j) & \text{if } j = k \end{aligned}$$

provided that m_j and m_k are any two zeros of J_0 . Therefore, we require that the m_j 's all be zeros of J_0 . This also ensures that $n = 0$ for $r = R$ ($\rho = 1$).

We have, then,

$$\begin{aligned} \frac{\partial}{\partial \zeta} \left\{ \sum_{j=1}^{\infty} B_j \exp \left(- \sqrt{m_j^2 - (2.4)^2} \zeta \right) J_0(m_j \rho) \right\}_{\zeta=0} &= \begin{cases} \frac{\beta n_p Q}{\rho_p} & \rho \leq \rho_p \\ 0 & \rho \geq \rho_p \end{cases} \\ - \sum_{j=1}^{\infty} B_j \sqrt{m_j^2 - 5.76} J_0(m_j \rho) &= \begin{cases} \frac{\beta n_p Q}{\rho_p} & \rho \leq \rho_p \\ 0 & \rho \geq \rho_p \end{cases} \end{aligned}$$

Multiply both sides by $\rho J_0(m_k \rho) \partial \rho$ and integrate from zero to unity.

$$\frac{\beta n_p Q}{\rho_p} \int_0^{\rho_p} \rho J_0(m_k \rho) d\rho = - \frac{B_k \sqrt{m_k^2 - (2.4)^2} J_1^2(m_k)}{2}$$

$$\frac{1}{m_k^2} \frac{\beta n_p Q}{\rho_p} [m_k \rho_p J_1(m_k \rho_p)] = - \frac{B_k \sqrt{m_k^2 - (2.4)^2} J_1^2(m_k)}{2}$$

$$B_k = - \frac{2 \beta n_p Q J_1(m_k \rho_p)}{m_k \sqrt{m_k^2 - (2.4)^2} J_1^2(m_k)} \quad k \neq 1$$

The coefficient of the term for $k = 1$ (for which $\sqrt{m_k^2 - (2.4)^2} = 0$) can be determined by noting that this term has no ζ -dependence, and a ρ -dependence of $J_0(2.4\rho)$. Its coefficient must be the value of n on the axis an infinite distance from the probe, which is n_0 , the axial density in the unperturbed plasma:

$$n(\rho, \zeta) = n_0 J_0(2.4\rho) - 2 \beta n_p Q \sum_{j=2}^{\infty} \frac{\exp\left(-\sqrt{m_j^2 - (2.4)^2} \zeta\right) J_1(m_j \rho_p) J_0(m_j \rho)}{m_j \sqrt{m_j^2 - (2.4)^2} J_1^2(m_j)}$$

which is the desired solution.

3. Alternative Determination of Plasma Potential Perturbation

Our purposes here are to determine the integral

$$\frac{n_0 Q_e (V_e + V_i)}{(1 + \beta Q) r_p} \int_0^{\infty} \frac{dz}{n},$$

using a different function for n , and to show that its form is not greatly different from the result obtained in Eq. 31. Therefore the value of the integral is not too sensitive to the exact shape of the function chosen for n .

It must be pointed out that, since n approaches a constant value, the integral will diverge. The physical reason for this is that the integral represents the integral of the electric field in the plasma which is required to draw the electron probe current and hence will ultimately become the integral of $J_e / \mu_e n_e$. However, we note that in the positive column of the sort of discharge which we are discussing here, an increased current requires a higher electron density, but the electric field remains approximately constant, or even decreases. In the actual physical case, the drawing of probe current requires that the axial density at infinity be higher than in the unperturbed case. Since our assumptions of $r_p \ll R$ imply that probe current will be very much less than tube current, we shall neglect the disturbance of axial electron density at infinity. The divergence of the integral will be overcome by subtracting the electric field required to draw the probe current at constant axial density, n_0 , and write

$$\frac{n_o Q_e (V_e + V_i)}{(1 + \beta Q) r_p} \int_0^\infty \left(\frac{1}{n} - \frac{1}{n_o} \right) dz.$$

This particular problem was avoided before by integrating between the limits n_p and n_o ; this took us not to infinity but to $z = \beta r_p$ with the function chosen.

Assuming that $n = n_o \left(1 - \frac{\beta Q}{1 + \beta Q} e^{-z/\beta r_p} \right)$, we obtain

$$I = \frac{Q_e (V_e + V_i)}{(1 + \beta Q) r_p} \int_0^\infty \frac{1}{1 - \frac{\beta Q}{1 + \beta Q} e^{-z/\beta r_p}} - 1$$

$$I = \frac{Q_e (V_e + V_i)}{(1 + \beta Q) r_p} \int_0^\infty \frac{\left(\frac{\beta Q}{1 + \beta Q} \right) e^{-z/\beta r_p}}{1 - \frac{\beta Q}{1 + \beta Q} e^{-z/\beta r_p}} dz$$

$$I = \frac{\beta Q_e (V_e + V_i)}{(1 + \beta Q)} \int_{u=0}^\infty \frac{Ae^{-u} du}{1 - Ae^{-u}}$$

$$= \frac{\beta Q_e (V_e + V_i)}{1 + \beta Q} \ln \{ 1 - Ae^{-u} \}_0^\infty$$

$$= - \frac{\beta Q_e (V_e + V_i)}{1 + \beta Q} \ln \left\{ 1 - \frac{\beta Q}{1 + \beta Q} \right\}$$

$$= \frac{\beta Q_e (V_e + V_i)}{(1 + \beta Q)} \ln (1 + \beta Q).$$

APPENDIX III

CALCULATION OF ION ENERGY AT THE SHEATH BOUNDARY

Starting with the fact that

$$V_{is} \approx \frac{V_i - V_g}{2} - E_p \lambda_i,$$

expressing

$$\mu_i = \frac{e}{m_i v_c} = \frac{e}{m_i} \frac{\lambda_i}{\sqrt{3eV_i/m_i}}$$

$$\lambda_i = \mu_i \sqrt{\frac{3V_i m_i}{e}},$$

and, from Eq. 11, obtaining

$$\begin{aligned} E_p &= -V_e \frac{1}{n} \frac{dn}{dr} \bigg|_{r_p} + \frac{Q_e(V_e + V_i)}{r_p} \\ &= -\frac{QV_e}{r_p} + \frac{Q_e(V_e + V_i)}{r_p} \\ &= \frac{Q_e V_i - Q_i V_e}{r_p}, \end{aligned}$$

we have

$$V_{is} \approx \frac{V_i - V_g}{2} + \left(\frac{Q_i V_e - Q_e V_i}{r_p} \right) \left(\mu_i \sqrt{\frac{3V_i m_i}{e}} \right).$$

Replacing the Q's by their values, we have

$$\begin{aligned} V_{is} \approx \frac{V_i - V_g}{2} + \left\{ \frac{V_e}{V_e + V_i} \sqrt{\frac{eV_{is}}{2\pi m_i}} \exp(-V_s/V_{is}) \right. \\ \left. - \frac{V_i}{V_e + V_i} \frac{\mu_i}{\mu_e} \sqrt{\frac{eV_{es}}{2\pi m_e}} \exp(V_s/V_{es}) \right\} \sqrt{\frac{3V_i m_i}{e}}. \end{aligned}$$

Here, the two exponential terms are retained only when they are less than 1.

Multiplying through, we have

$$V_{is} \approx \frac{V_i - V_g}{2} + \frac{V_e}{V_e + V_i} \sqrt{\frac{3V_{is} V_i}{2\pi}} \exp(-V_s/V_{is}) - \frac{V_i}{V_e + V_i} \frac{\mu_i}{\mu_e} \sqrt{\frac{3V_i V_i m_i}{2\pi m_e}} \exp(V_s/V_{es}).$$

In the vicinity of zero sheath potential the second term is small in comparison to the first; hence, for the moment, we assume that $\sqrt{V_{is} V_i} \approx V_i$.

$$V_{is} \approx \frac{V_i - V_g}{2} + \frac{V_e}{V_e + V_i} \left(\sqrt{\frac{3}{2\pi}} \right) V_i.$$

For $V_i \approx V_g \ll V_e$,

$$V_{is} \approx \sqrt{\frac{3}{2\pi}} V_i \approx \frac{V_i}{\sqrt{2}}$$

For $V_i \approx V_e \gg V_g$,

$$V_{is} = \frac{V_i}{2} + \frac{V_i}{2} \sqrt{\frac{3}{2\pi}} \approx \frac{V_i}{2} \left(1 + \frac{1}{\sqrt{2}} \right).$$

Therefore, it is reasonable to say that V_{is} , the ion temperature at the sheath edge, is approximately equal to the ion temperature in the undisturbed plasma.

Acknowledgment

It is a pleasure to acknowledge the helpful advice of Professor Francis Bitter. I am also indebted to Professor William P. Allis for several very illuminating discussions and for the suggestion that the ion temperature be retained explicitly in the expressions.

References

1. I. Langmuir and H. M. Mott Smith, Gen. Elec. Rev. 27, 499; 538; 616; 762; 810 (1924).
2. R. J. Bickerton and A. von Engel, Proc. Phys. Soc. (London) 69B, 468 (1956).
3. B. Bertotti, Phys. Fluids 4, 1047 (1961).
4. W. P. Allis and S. J. Buchsbaum, Notes on Plasma Dynamics, Summer Session, M.I.T., 1959 (unpublished).
5. B. Davydov and L. Zmanovskaja, Tech. Phys. (U.S.S.R.) 3, 715 (1936).
6. R. L. F. Boyd, Proc. Phys. Soc. (London) 64B, 795 (1951).
7. W. Schottky, Physik. Z. 25, 635 (1924).
8. L. Tonks and I. Langmuir, Phys. Rev. 34, 876 (1929).
9. J. M. Anderson, J. Appl. Phys. 31, 511-515 (1960).
10. D. G. Bills, R. B. Holt, and B. T. McClure, J. Appl. Phys. 33, 29 (1962).
11. G. Medicus, J. Appl. Phys. 27, 1242 (1956).
12. W. Verweij, Philips Research Rept. Supplement No. 2, 1961; Physica 25, 980 (1959).
13. J. F. Waymouth, Pulse technique for probe measurements in gas discharges, J. Appl. Phys. 30, 1404 (1959).
14. J. M. Anderson, General Electric Report 58-RL-2010, August 1958; cf. ref. 9 for the published version of this report.